Math 113: Final Exam

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Name :

Student ID Number:

Instructions: This is a closed-book test. Each problem is worth 20 points. Read the questions carefully, and show all your work. All work should be done on the exam paper. Additional white paper is available if needed. Good luck.

Problem	Score
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Total	

(1) (a) Let G_1 and G_2 be groups. Define the **direct product** $G_1 \times G_2$.
(b) Generalize the notion for groups $G_1, ..., G_n$. Prove that $G_1 \times G_2 \times ... \times G_n$ is a group.

(2) Let G be a finite cyclic group of order n. What are the generators of G? What are the automorphisms of G? What are the subgroups of G? Prove your answers.

(3) S_n is the group of permutations of $\{1, 2, ..., n\}$. A transposition (rs) interchanges two elements and keeps the rest fixed. Show the transpositions generate the whole permutation group S_n.

(4) (a) Let G be a group, H a subgroup, and x an element in G. Define the **center** of G, the **normalizer** of H, and the **centralizer** of x in G.
(b) What is the relationship between an element in the center and its centralizer? What is the relationship between H and its normalizer?

(5) Let $\phi : G \to G'$ be a surjective group homomorphism. State and prove the First Isomorphism Theorem.

(6) Classify groups of order 39.

(7) Let G be a finite group. Let p be the smallest prime which divides the order of G. Let H be a subgroup of index p. Show that H is normal.

(8) Let R be the set of rational numbers whose denominator is a power of 2, (e.g. $\frac{1}{8}, \frac{41}{2}, 11$). Show that R is a ring. What are the units of R? What are the prime elements of R? Is R a unique factorization domain? Explain your answers.

(9) (a) Define a **nilpotent** element of a ring R.
(b) Prove that if x is a nilpotent element, then x is in every prime ideal of R.

(10) Prove that M is a maximal ideal in R if and only if R/M is a field.

(11) (a) Prove or disprove: $\mathbb{Z}[x]$ is a principal ideal domain (PID).
(b) Prove or disprove: $F[x]$ is a PID, where F is a field.
(c) Prove or disprove: $F[x_1, x_2, ..., x_n]$ is a PID, where F is a field.

(12) (a) Determine whether the following polynomials are irreducible in $\mathbb{Q}[x]$ and explain your answers:
(i) $x^n - a$, where a is a square-free integer
(ii) $81x^4 + 3x + 1$.
(b) Describe the quotient ring $\mathbb{Z}[i]/(2 + 5i)$.

(13) (a) Define a **primitive polynomial**.

(b) Let f and g be primitive polynomials in $R[x]$, where R is a UFD. Prove that if f and g are degree one polynomials, then fg is primitive. (Prove directly, not as a corollary to another theorem.)