MATHEMATICS 170 — MIDTERM F03 EVANS

Problem 1. Start with the basic feasible vector $x^T = [0,1,1]$ and then use the *simplex tableau method* to solve

$$(P)\qquad \begin{cases} \text{minimize } x_1 + x_2 + 3x_3, \text{ subject to the constraints} \\ x_1 + x_3 = 1,\ x_2 + x_3 = 2,\ x_1 \geq 0, x_2 \geq 0, x_3 \geq 0. \end{cases}$$

In particular, demonstrate how to set up the initial tableau and then how to modify it by pivoting, to get an optimal solution.

Problem 2. The vector $x^T = [\frac{4}{3}, \frac{10}{3}, 0, 0]$ is an optimal solution of:

$$(P)\qquad \begin{cases} \text{minimize } 3x_1 + 2x_2, \text{ subject to the constraints} \\ 2x_1 + x_2 - x_3 = 6,\ x_1 + 2x_2 - x_4 = 8, \\ x_1 \geq 0, x_2 \geq 0, x_3 \geq 0, x_4 \geq 0. \end{cases}$$

Use the equilibrium equations to find an optimal solution $y^T = [y_1, y_2]$ of the dual problem (D).

Problem 3. Prove that if there exists a solution x of

$$(*)\qquad Ax = b,\ x \geq 0,$$

then there exists a basic solution.
(Hint: Take a solution x of $(*)$ with the fewest number of nonzero components. If the columns $\{a^\alpha, a^\beta, \ldots, a^\sigma\}$ of A corresponding to the nonzero entries in x are dependent, we can find $\{\theta_\alpha, \theta_\beta, \ldots, \theta_\sigma\}$, not all zero, such that $\theta_\alpha a^\alpha + \theta_\beta a^\beta + \cdots + \theta_\sigma a^\sigma = 0$. Finish the proof from here.)

Problem 4. Let A be an $m \times n$ matrix.
(a) State the *Farkas alternative* (ii) to the assertion

$$(i)\qquad Ax = b,\ x \geq 0 \quad \text{has a solution } x.$$

(b) Use the Separating Hyperplane Theorem to prove that either (i) or (ii) holds, but not both.
(Hint: You may assume that the finite cone $C = \{Ax \mid x \geq 0\}$ is closed and convex.)

Problem 5. Suppose A is a symmetric $n \times n$ matrix; that is, $A^T = A$. Consider the linear programming problem:

$$(P)\qquad \text{minimize } b \cdot x, \quad \text{subject to } Ax = b,\ x \geq 0.$$

Show that any feasible x is in fact optimal.