George M. Bergman
70 Evans Hall

Spring 2004, Math 185, Sec. 1

Second Midterm

2 April, 2004
10:10-11:00

1. (40 points, 10 points each.) For each of the items listed below, either *give an example* with the properties stated, or give a brief reason why *no such example exists.*

If you give an example, you do *not* have to prove that it has the property stated; however, your examples should be specific; i.e., even if there are many objects of a given sort, name a particular one. If you give a reason why no example exists, don't worry about giving a detailed proof; the key relevant fact will suffice.

(a) An interval $[a, b]$, and two closed paths $\gamma_1, \gamma_2 : [a, b] \to \mathbb{C} \setminus \{0\}$, such that γ_1 and γ_2 are not homotopic as paths in $\mathbb{C} \setminus \{0\}$, but are homotopic as paths in $\mathbb{C}$.

(b) An interval $[a, b]$, and two closed paths $\gamma_1, \gamma_2 : [a, b] \to \mathbb{C} \setminus \{0\}$, such that γ_1 and γ_2 are not homotopic as paths in $\mathbb{C}$, but are homotopic as paths in $\mathbb{C} \setminus \{0\}$.

(c) A differentiable function f on the punctured disk $\{z \in \mathbb{C} \mid 0 < |z| < 1\}$ which has a pole of order 2 at $z = 0$, and satisfies $\mathrm{res}(f, 0) = 5$.

(d) A differentiable function f on the punctured disk $\{z \in \mathbb{C} \mid 0 < |z| < 1\}$ which has an essential singularity at $z = 0$.

2. (10 points) Suppose f is a differentiable function on a disc $N_R(z_0)$, C_r is a circle of radius $r < R$ about z_0, and w is a point inside C_r. Give a formula (the Cauchy Integral Formula) by which $f(w)$ can be computed from the values of f on C_r.

3. (15 points) Suppose a function f differentiable in a punctured disc $D = N_R(z_0) \setminus \{z_0\}$ has a simple pole at z_0. Show that f does not have an antiderivative in D.

4. (15 points) Let r be a positive real number, and f a differentiable function on the domain $D = \{z \in \mathbb{C} \mid |z| > r\}$. Show that if f is bounded on D, then $\lim_{z \to \infty} f(z)$ exists. (We recall that saying the limit exists means that there is some complex number L such that $\lim_{z \to \infty} f(z) = L$, i.e., such that for every $\varepsilon > 0$ there exists an $R > 0$ such that for all z with $|z| > R$, one has $|f(z) - L| < \varepsilon$.)

Hint: Think about the kind of singularity f can have at ∞.

5. (20 points = 10+3+5+2) Let $\rho_1 \neq \rho_2$ be complex numbers, and C a circle in $\mathbb{C}$, oriented counterclockwise.

(a) Compute the residues of $1/((z - \rho_1)(z - \rho_2))$ at ρ_1 and at ρ_2.

For the next three parts, give the value of $\int_C 1/((z - \rho_1)(z - \rho_2))\, dz$ if ...

(b) both ρ_1 and ρ_2 are inside C:

(c) ρ_1 is inside C and ρ_2 is outside C:

(d) both ρ_1 and ρ_2 are outside C: