On the Normality of Numbers in Different Bases

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1 Introduction

We ask whether normality to one base is related to normality to another. Maxfield in 1953 proved that a real number is normal to a base exactly when it is normal to every base multiplicatively dependent to that base (two numbers are multiplicatively dependent when one is a rational power of the other.) Schmidt (1961/1962) showed that this is the only restriction on the set of bases to which a real number can be normal. He proved that for any given set of bases, closed under multiplicative dependence, there are real numbers normal to every base from the given set and not normal to any base in its complement. This result, however, does not settle the question of whether the discrepancy functions for different bases for which a real number is normal are pairwise independent. Nor does it answer whether the set of bases for which a real number is normal plays a distinguished role among its other arithmetical properties.

We pose these problems by means of mathematical logic and descriptive set theory. The set of real numbers that are normal to a least one base is located in the fourth level of the Borel hierarchy. Similarly, the set of indices for computable real numbers that are normal to at least one base is located at the fourth level of the arithmetic hierarchy. In Theorem 1 we show that from both points of view, the property that a real number is normal to at least one base is complete at the fourth level ($\mathbf{\Sigma}^0_4$ and Σ^0_4, respectively). This result settles a question in Bugeaud (2012) and confirms a conjecture of A. Ditzen (see Ki and Linton, 1994). We obtain the result by first establishing in Theorem 2, that for any set at the third level of the arithmetic hierarchy (Π^0_3), there is a computable real number which is normal exactly to the bases multiplicatively dependent to elements of that set. Theorem 3 exhibits a fixed point: for any property of bases expressed at the third level of the arithmetic hierarchy (Π^0_3) and closed under multiplicative dependence, there is a real number ξ such that the bases which satisfy the property relative to ξ are exactly those for which ξ is normal.

Theorem 4 shows that the discrepancy functions for different bases can go to zero independently. We construct absolutely normal real numbers such that their discrepancy functions for a given base s converge to zero arbitrarily slowly and such that their discrepancies for all the bases multiplicatively independent to s are eventually dominated by a single computable bound. In contrast, the real numbers constructed by Schmidt (1961/1962) are not normal to a given base s and the discrepancy functions for all bases multiplicatively independent to s converge to zero at a prescribed rate. With a different proof, Brown, Moran, and Pearce (1985) extended Schmidt's result and then Moran and Pearce (1988) gave explicit bounds for the rate obtained with their method. In our construction the nonconforming behavior of the constructed real number with respect to base s appears even though it is normal to base s.

Theorem 5 sharpens Theorem 1 in Schmidt (1961/1962). We construct a real number that is normal for all elements in a given set and denies even simple normality to all other elements, addressing an issue raised in Brown et al. (1985).

Normality is an almost-everywhere property of the real numbers: the set of normal numbers has Lebesgue measure one. Normality in some bases and not all of them is also an almost-everywhere property, albeit not in the sense of Lebesgue. Consider the Cantor set C_s obtained by omitting the last digit (or two) in the base s expansions of real numbers (s greater than 2). Clearly, no element of C_s is simply normal to base s. However, viewed from the perspective of the uniform measure on this Cantor set, Schmidt (1960) shows that the subset of C_s whose elements are normal to every base r multiplicatively independent to s has measure one.

Our focus is on constructing real numbers and maintaining independent control over their discrepancy functions for multiplicatively independent bases. Since almost every element of C_s is normal to base r, almost every sufficiently long finite initial segment of a real in C_s has small discrepancy from normal in base r. It is our task to convert this observation into methods of constructing real numbers by iteratively extending their expansions in various bases. The first part of our task is to give computable bounds on discrepancy and estimates on how quickly discrepancy for base r decreases almost everywhere in C_s. The second part is to convert these finitary bounds into modules for constructions. The typical module lowers discrepancy in bases r from a finite set R and increases discrepancy in a multiplicatively independent base s. It is important that the estimates on discrepancy be applicable in any basic open neighborhood in C_s so that the modules can be used as any finite point in the construction.

2 Theorems

Notation. A *base* is an integer greater than or equal to 2. For a real number ξ, we use $\{\xi\}$ to denote its fractional part. We write $\vec{\xi}$ to denote a sequence and ξ_j to denote the jth element of $\vec{\xi}$. If $\vec{\xi}$ is finite with N elements we write it as $(\xi_1, \ldots, \xi_N)$. For a subinterval I of $[0, 1]$, $\mu(I)$ is its measure, equivalently its length. For a finite set S, $\#S$ is its cardinality. We often drop the word *number* and just say *a real* or *a rational* or *an integer*.

We recall the needed definitions and then state our results. The usual presentation of the property of normality to a given base for a real number is in terms of counting occurrences of blocks of digits in its expansion in that base (Bugeaud (2012); Kuipers and Niederreiter (2006)). Absolute normality is normality to all bases. We define normality in terms of discrepancy. See either of the above references for a proof of Wall's Theorem, which establishes the equivalence.

Definition 2.1. The *discrepancy* of a sequence $\vec{\xi} = (\xi_1, \ldots, \xi_N)$ of real numbers in the unit interval is

$$D(\vec{\xi}) = \sup_{0 \leqslant u < v \leqslant 1} \left| \frac{\#\{n : 1 \leqslant n \leqslant N, u \leqslant \xi_n < v\}}{N} - (v - u) \right|.$$

When we refer to a sequence by specifying its elements, we will write $D(\xi_1, \ldots, \xi_N)$, rather than $D((\xi_1, \ldots, \xi_N))$.

Definition 2.2. Let r be a base. A real number ξ is *normal to base* r if and only if $\lim_{N \to \infty} D(\{r^j \xi\} : 0 \leqslant j < N) = 0$. *Absolute normality* is normality to every base.

A formula in the language of arithmetic is Π^0_0 and Σ^0_0 if all of its quantifiers are bounded. It is Σ^0_{n+1} if it has the form $\exists x\, \theta$ where θ is Π^0_n and it is Π^0_{n+1} if it has the form $\forall x\, \theta$ where θ is Σ^0_n. A subset A of $\mathbb{N}$ is Σ^0_n (respectively, Π^0_n) if there is a Σ^0_n (respectively, Π^0_n) formula φ such that for all n, $n \in A$ if and only if $\varphi(n)$ is true. A Σ^0_n subset A of the natural numbers is Σ^0_n-complete if there is a computable function f mapping Σ^0_4 formulas to natural numbers such that for all φ, φ is true in the natural numbers if and only if $f(\varphi) \in A$.

The Borel hierarchy for subsets of $\mathbb{R}$ with the usual topology states that a set A is $\mathbf{\Sigma}^0_1$ if and only if A is open and A is $\mathbf{\Pi}^0_1$ if and only if A is closed. A is $\mathbf{\Sigma}^0_{n+1}$ if and only if it is a countable union of $\mathbf{\Pi}^0_n$ sets and A is $\mathbf{\Pi}^0_{n+1}$ if and only if it is a countable intersection of $\mathbf{\Sigma}^0_n$ sets. By an important theorem, a $\mathbf{\Sigma}^0_n$ subset of $\mathbb{R}$ is $\mathbf{\Sigma}^0_n$-complete if and only if it is not $\mathbf{\Pi}^0_n$.

Theorem 1. *(1) The set of indices for computable real numbers which are normal at least one base is Σ^0_4-complete. (2) The set of real numbers that are normal to at least one base is $\mathbf{\Sigma}^0_4$-complete.*

Remark 2.3. A routine extension of the proof shows that the set of real numbers which are normal to infinitely many bases is $\mathbf{\Pi}^0_5$-complete. Expressed in terms of the complement, the set of real numbers which are normal to only finitely many bases is $\mathbf{\Sigma}^0_5$-complete.

Let M be the set of minimal representatives of the multiplicative dependence equivalence classes. Our proof of Theorem 1 relies on the following.

Theorem 2. *For any Π^0_3 subset R of M there is a computable real number ξ such that for all r in M, $r \in R$ if and only if ξ is normal to base r. Furthermore, ξ is computable uniformly in the Π^0_3 formula that defines R.*

Theorem 3 exhibits a fixed point: the real ξ appears in the Π^0_3 definition of its input set. It asserts that the set of bases for which ξ is normal can coincide with any other property of elements of M definable by a Π^0_3 formula relative to ξ. Thus, the set of bases for normality can be arbitrary, nothing distinguishes it from other Π^0_3 predicates on M. As a subset of $\mathbb{N}$ its only distinguishing feature is that it is closed under multiplicative dependence.

Theorem 3. *For any Π^0_3 formula φ there is a computable real number ξ such that for any base $r \in M$, $\varphi(\xi, r)$ is true if and only if ξ is normal to base r.*

Theorem 4 illustrates the independence between the discrepancy functions for multiplicatively independent bases by exhibiting an extreme case, that all but one of the bases behave predictably and the other is arbitrarily slow.

Theorem 4. *Fix a base s. There is a computable function $f : \mathbb{N} \to \mathbb{Q}$ monotonically decreasing to 0 such that for any function $g : \mathbb{N} \to \mathbb{Q}$ monotonically decreasing to 0 there is an absolutely normal real number ξ whose discrepancy for base s eventually dominates g and whose discrepancy for each base multiplicatively independent to s is eventually dominated by f. Furthermore, ξ is computable from g.*

Remark 2.4. The proof Theorem 4 can be adapted produce other contrasts in behavior between multiplicatively independent bases. We give two examples.

(1) Let s be a base. There is a computable function $f : \mathbb{N} \to \mathbb{Q}$ monotonically decreasing to 0 such that for any function $g : \mathbb{N} \to \mathbb{N}$, there is an absolutely normal real number ξ such that its discrepancy for s satisfies for all n there is an $N > g(n)$ such that $D(\{s^j\xi\} : 0 \leqslant j < N) > 1/n$

and its discrepancies for bases multiplicatively independent to s are eventually bounded by f. Furthermore, ξ is computable from any real number ρ which can computably approximate g.

(2) Let s and r be multiplicatively independent bases. There is a computable absolutely normal number ξ such that

$$\limsup_{N\to\infty} \frac{D(\{s^j\xi\} : 0 \leqslant j < N)}{D(\{r^j\xi\} : 0 \leqslant j < N)} = \limsup_{N\to\infty} \frac{D(\{r^j\xi\} : 0 \leqslant j < N)}{D(\{s^j\xi\} : 0 \leqslant j < N)} = \infty.$$

Remark 2.5. There is a computable function $f : \mathbb{N} \to \mathbb{Q}$ monotonically decreasing to 0 such that the discrepancy of almost every real number is eventually dominated by f. In contrast, there is no computable function which dominates the discrepancy of all the computable absolutely normal numbers.

Finally, we state the improvement of Theorem 1 of Schmidt (1961/1962), asserting simple normality in the conclusion.

Definition 2.6. Let N be a positive integer. Let $\xi_1, \ldots, \xi_N$ be real numbers in $[0, 1]$. Let F be a family of subintervals. The discrepancy of $\xi_1, \ldots, \xi_N$ for F is

$$D(F, (\xi_1, \ldots, \xi_N)) = \sup_{I \in F} \left| \frac{\#\{n : \xi_n \in I\}}{N} - \mu(I) \right|.$$

Definition 2.7. Let r be a base and let ξ be a real number. Let F be the set of intervals of the form $[a/r, (a+1)/r)$, where a is an integer $0 \leqslant a < r$. ξ is *simply normal to base* r if $\lim_{N\to\infty} D(F, (\{r^j\xi\} : 0 \leqslant j < N)) = 0$.

Theorem 5. *Let R be a set of bases closed under multiplicative dependence. There are real numbers normal to every base from R and not simply normal to any base in its complement. Furthermore, such a real number can be obtained computably from R.*

3 Lemmas

3.1 On Uniform Distribution of Sequences

Lemma 3.1. *Let ϵ be a real number strictly between* 0 *and* 1*. Let F_ϵ be the family of semi-open intervals $B_a = [a/\lceil 3/\epsilon \rceil, (a+1)/\lceil 3/\epsilon \rceil)$, where a is an integer $0 \leqslant a < \lceil 3/\epsilon \rceil$. For any sequence $\vec{\xi}$ and any N, if $D(F_\epsilon, \vec{\xi}) < (\epsilon/3)^2$ then $D(\vec{\xi}) < \epsilon$.*

Proof. Let $\vec{\xi}$ be a sequence of real numbers of length N such that $D(F_\epsilon, \vec{\xi})$ is less than $(\epsilon/3)^2$. Let I be any semi-open subinterval of $[0, 1]$. Denote $\lceil 3/\epsilon \rceil$ by n. The number of B_a with nonempty intersection with I is less than or equal to $\lceil n\mu(I) \rceil$. For each $B_a \in F_\epsilon$, $\#\{\xi_n : \xi_n \in B_a\}$ is less than or equal to $(1/n + \epsilon^2/9)N$. Thus, by the definition of n,

$$\frac{1}{N}\#\{\xi_n : \xi_n \in I\} \leqslant \frac{1}{N}\lceil n\mu(I) \rceil (1/n + \epsilon^2/9)N \leqslant \mu(I) + \epsilon.$$

Similarly, $\frac{1}{N}\#\{\xi_n : \xi_n \in I\} \geqslant \mu(I) - \epsilon$. □

Remark 3.2. In Lemma 3.1, F_ϵ can be replaced by any partition of $[0, 1]$ into subintervals of equal length, each of length at most $\epsilon/3$.

We record the next three observations without proof.

Lemma 3.3. *Suppose that ϵ is a positive real, $\vec{\xi}$ is a sequence of length N and that $D(\vec{\xi}) < \epsilon$. For any sequence $\vec{\nu}$ of length n with $n < \epsilon N$, for all $k \leqslant n$, $D(\nu_1 \ldots, \nu_k, \xi_1, \ldots, \xi_N) < 2\epsilon$ and $D(\xi_1, \ldots, \xi_N, \nu_1 \ldots, \nu_k) < 2\epsilon$.*

Lemma 3.4. *Let $\vec{\xi}$ be a sequence of real numbers, ϵ a positive real and $(b_m : 0 \leqslant m < \infty)$ an increasing sequence of positive integers. Suppose that there is an m_0 such that for all $m > m_0$, $b_{m+1} - b_m \leqslant \epsilon b_m$ and $D(\xi_j : b_m < j \leqslant b_{m+1}) < \epsilon$. Then $\lim_{N\to\infty} D(\vec{\xi}) \leqslant 2\epsilon$.*

Lemma 3.5. *Let m be a positive integer and I a semi-open interval. Suppose $\vec{\xi}$ is a sequence of real numbers of length N such that $N \geqslant \lceil 2m/\mu(I) \rceil$ and for all j with $m \leqslant j \leqslant N$, $\xi_j \notin I$. Then, $D(I, \vec{\xi}) \geqslant \mu(I)/2$.*

Notation. We let $e(x)$ denote $e^{2\pi i x}$.

Theorem 3.6 (Weyl's Criterion (see Bugeaud, 2012))**.** *A sequence $(\xi_n : n \geqslant 1)$ of real numbers is uniformly distributed modulo one if and only if for every non-zero t, $\displaystyle\lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} e(t\xi_n) = 0$.*

Theorem 3.7 (**LeVeque's Inequality** (see Kuipers and Niederreiter, 2006, Theorem 2.4))**.** *Let $\vec{\xi} = (\xi_1, \ldots, \xi_N)$ be a finite sequence. Then, $D(\vec{\xi}) \leqslant \Big(\frac{6}{\pi^2} \sum_{h=1}^{\infty} \frac{1}{h^2} \Big| \frac{1}{N} \sum_{j=1}^{N} e(h\xi_j)\Big|^2\Big)^{\frac{1}{3}}$.*

Lemma 3.8. *For any positive real ϵ there is a finite set T of integers and a positive real δ such that for any $\vec{\xi} = (\xi_1, \ldots, \xi_N)$, if for all $t \in T$, $\displaystyle\frac{1}{N^2} \Big|\sum_{j=1}^{N} e(t\xi_j)\Big|^2 < \delta$ then $D(\vec{\xi}) < \epsilon$. Furthermore, such T and δ can be computed from ϵ.*

Proof. By LeVeque's Inequality, $D(\vec{\xi}) \leqslant \Big(\frac{6}{\pi^2} \sum_{h=1}^{\infty} \frac{1}{h^2} \Big| \frac{1}{N} \sum_{j=1}^{N} e(h\xi_j)\Big|^2\Big)^{\frac{1}{3}}$. Note that $\Big|\frac{1}{N} \sum_{j=1}^{N} e(h\xi_j)\Big|^2 \leqslant 1$. Hence, for each h,

$$\sum_{h=m+1}^{\infty} \frac{1}{h^2} \Big| \frac{1}{N} \sum_{j=1}^{N} e(h\xi_j)\Big|^2 \leqslant \sum_{h=m+1}^{\infty} \frac{1}{h^2} \leqslant \int_{m+1}^{\infty} x^{-2} dx \leqslant \frac{1}{m+1}.$$

Assume $\displaystyle\frac{1}{N^2} \Big|\sum_{j=1}^{N} e(t\xi_j)\Big|^2 < \delta$ for all positive integers t less than or equal to m. Then,

$$\sum_{h=1}^{m} \frac{1}{h^2} \Big| \frac{1}{N} \sum_{j=1}^{N} e(h\xi_j)\Big|^2 + \sum_{h=m+1}^{\infty} \frac{1}{h^2} \Big| \frac{1}{N} \sum_{j=1}^{N} e(h\xi_j)\Big|^2 \leqslant \sum_{h=1}^{m} \frac{1}{h^2}\delta + \frac{1}{m+1} \leqslant \delta m + \frac{1}{m+1}.$$

To ensure $D(\vec{\xi}) < \epsilon$ it is sufficient that $(6/\pi^2)\big(\delta m + (1/m+1)\big)^{\frac{1}{3}} < \epsilon$. This is obtained by setting $\delta m < (1/2)(\epsilon^3\pi^2/6)$ and $1/(m+1) < (1/2)(\epsilon^3\pi^2/6)$. Let $m = \lceil 12/(\epsilon^3\pi^2) \rceil$, $T = \{1, 2, \ldots, m\}$ and $\delta = (\epsilon^3\pi^2)/(24m)$. $\square$

3.2 On Normal Numbers

Notation. We use $\langle b; r\rangle$ to denote $\lceil b/\log r\rceil$, where log refers to natural logarithm. We say that a rational number $\eta \in [0,1]$ is s-adic when $\eta = \sum_{j=1}^{a} d_j s^{-j}$ for digits d_j in $\{0,\ldots,s-1\}$. In this case, we say that η has precision a. We use $\mathcal{L}(s,k)$ to denote sequences in the alphabet $\{0,\ldots,s-1\}$ of length k. For a sequence w, we write $|w|$ to denote its length. When $1 \leqslant i \leqslant j \leqslant |w|$, we call $(w_i,\ldots,w_j)$ a block of w. The number of occurrences of the block u in w is $occ(w,u) = \#\{i : (w_i,\ldots,w_{i+|u|-1}) = u\}$.

Lemma 3.9. *Let s and r be bases, a be a positive integer and ϵ be a real between 0 and 1. There is a finite set of intervals F and a positive integer ℓ_0 such that for all $\ell \geqslant \ell_0$ and all ξ_0, if $\xi \in [\xi_0, \xi_0 + s^{-\langle a+\ell;s\rangle})$ and $D(F, (\{r^j\xi_0\} : \langle a;r\rangle < j \leqslant \langle a+\ell;r\rangle)) < (\epsilon/10)^4$ then $D(\{r^j\xi\} : \langle a;r\rangle < j \leqslant \langle a+\ell;r\rangle) < \epsilon$. Furthermore, ℓ_0 and F can be taken as computable functions of r and ϵ.*

Proof. Let F_ϵ be as in Lemma 3.1 and let I be an interval in F_ϵ. Let n denote $\lceil 100/\epsilon^2\rceil$. Let F be the set of semi-open intervals $B_c = [c/n, (c+1)/n)$, where $0 \leqslant c < n$. For the sake of computing ℓ_0, consider $b > a$, ξ and ξ_0 such that $\xi \in [\xi_0, \xi_0 + s^{-\langle b;s\rangle})$. Assume $D(F, (\{r^j\xi_0\} : \langle a;r\rangle < j \leqslant \langle b;r\rangle)) < (\epsilon/10)^4$.

Note that for all j less than $\langle b;r\rangle - \log n/\log r - 1$, we have $r^j s^{-\langle b;s\rangle} < 1/n$. Hence, for all but the last $\lceil \log n/\log r\rceil + 2$ values of j, $|r^j\xi_0 - r^j\xi| < 1/n$. Let C be the set of intervals B_c such that either B_c or B_{c+1} has non-empty intersection with I. If j is less than $\langle b;r\rangle - \log n/\log r - 1$ then $\{r^j\xi\} \in I$ implies that $\{r^j\xi_0\} \in \cup C$. Observe that $\#C \leqslant \lceil n\mu(I)\rceil + 2$. The fraction

$$\frac{1}{\langle b;r\rangle - \langle a;r\rangle}\#\{j : \langle a;r\rangle < j < \langle b;r\rangle - \log n/\log r - 1 \text{ and } \{r^j\xi\} \in \cup C\}$$

is at most $\lceil n\mu(I) + 2\rceil(1/n + \epsilon^4/10^4)$. And by definition of n,

$$(n\mu(I)+3)(1/n+(\epsilon/10)^4) \leqslant \mu(I)+\lceil 100/\epsilon^2\rceil(\epsilon/10)^4+3/\lceil 100/\epsilon^2\rceil+3(\epsilon/10)^4 \leqslant \mu(I)+(1/2)(\epsilon/3)^2.$$

There are at most $\lceil \log n/\log r\rceil + 2$ remaining j, those for which $j \geqslant \langle b;r\rangle - \log n/\log r - 1$. Suppose that for each such j, $r^j\xi \in I$. Then,

$$\frac{\lceil \log n/\log r\rceil + 2}{\langle b;r\rangle - \langle a;r\rangle} \leqslant \frac{\log n/\log r + 3}{\langle b;r\rangle - \langle a;r\rangle} \leqslant \frac{\log\lceil 100/\epsilon^2\rceil + 3\log r}{b - a - \log r}.$$

Let ℓ_0 be $\left\lceil \log r + \frac{18}{\epsilon^2}\lceil \log\lceil 100/\epsilon^2\rceil + 3\log r\rceil\right\rceil$. For $b \geqslant a + \ell_0$, $\frac{\log\lceil 100/\epsilon^2\rceil+3\log r}{b-a-\log r} < (1/2)(\epsilon/3)^2$. A similar argument yields the same estimates for the needed lower bound. Then, for F_ϵ and any $b \geqslant a + \ell_0$, $D(F_\epsilon, (\{r^j\xi\} : \langle a;r\rangle < j \leqslant \langle b;r\rangle)) < (\epsilon/3)^2$. By applying Lemma 3.1, for any $\ell \geqslant \ell_0$, $D(\{r^j\xi\} : \langle a;r\rangle < j \leqslant \langle a+\ell;r\rangle) < \epsilon$. □

Definition 3.10. Fix a base s. The *discrete discrepancy* of $w \in \mathcal{L}(s,N)$ for a block of size ℓ is

$$C(\ell,w) = \max\left\{\left|\frac{occ(w,u)}{N} - \frac{1}{s^\ell}\right| : u \in \mathcal{L}(s,\ell)\right\}.$$

The next lemmas relate the discrete discrepancy of sequences in $w \in \mathcal{L}(s,N)$ to the discrepancy of their associated sequences of real numbers.

Lemma 3.11. *Let ϵ be a positive real, s a base, ℓ and N positive integers such that $s^\ell > 3/\epsilon$ and $N > 2\ell(3/\epsilon)^2$, and $w \in \mathcal{L}(s, N)$ such that $C(\ell, w) < \epsilon^2/18$. Then, $D(\{s^j \eta_w\} : 0 \leqslant j < N) < \epsilon$, where $\eta_w = \sum_{j=1}^{|w|} w_j s^{-j}$.*

Proof. Let ℓ be such that $s^{-\ell} < \epsilon/3$. Let F be the set of s-adic intervals of length $s^{-\ell}$. Any I in F has the form $[\eta_u, \eta_u + s^{-\ell})$, for some $u \in \mathcal{L}(s, \ell)$, and further, $\{s^j \eta_w\} \in I$ if and only if the block u occurs in w at position $j + 1$. Thus, we can count instances of $\{s^j \eta_w\} \in I$ by counting instances of u in w. Let N and w be given so that $N > 2\ell(3/\epsilon)^2$, $w \in \mathcal{L}(s, N)$ and $C(\ell, w) < (\epsilon/3)^2/2$. Then, for any $u \in \mathcal{L}(s, \ell)$, $\left|occ(w, u)/N - s^{-\ell}\right| < (\epsilon/3)^2/2$. For any $I \in F$, $\frac{1}{N}\#\left\{j : \{s^j \eta_w\} \in I \text{ and } 0 \leqslant j < N\right\} < s^{-\ell} + (\epsilon/3)^2/2 + (\ell - 1)/N < s^{-\ell} + (\epsilon/3)^2$. A similar count gives the analogous lower bound. Hence, $D(F, (\{s^j \eta_w\} : 0 \leqslant j < N)) < (\epsilon/3)^2$ and so $D(\{s^j \eta_w\} : 0 \leqslant j < N) < \epsilon$, by application of Lemma 3.1 and Remark 3.2. ☐

Lemma 3.12 (see Theorem 148, Hardy and Wright (2008))**.** *For any base s, for any positive integer ℓ and for any positive real numbers ϵ and δ, there is an N_0 such that for all $N \geqslant N_0$,*

$$\#\Big\{v \in \mathcal{L}(s, N) : C(\ell, v) \geqslant \epsilon\Big\} < \delta s^N.$$

Furthermore, N_0 is a computable function of s, ϵ and δ.

The next lemma is specific to base 2 and will be applied in the proof of Theorem 5.

Lemma 3.13. *Given a positive real number ϵ, there is an N_0 such that for all $N \geqslant N_0$,*

$$\#\Big\{v \in \mathcal{L}(2, N) : \frac{1}{2N}\#\{m : \{2^m \eta_v\} \in [0, 1/2)\} \geqslant 5/8\Big\} > (1 - \epsilon)2^N$$

where for $v = (v_1, \ldots, v_N) \in \mathcal{L}(2, N)$, $\eta_v = \sum_{j=1}^N v_j 4^{-j}$. Furthermore, N_0 is a computable function of ϵ.

Proof. By Lemma 3.12, for any positive δ there is an N_0 such that for all $N \geqslant N_0$,

$$\#\Big\{v \in \mathcal{L}(2, N) : C(1, v) \leqslant \delta\Big\} \geqslant (1 - \epsilon)2^N.$$

Thus, for $(1-\epsilon)2^N$ many v, $\left|\dfrac{\#\{n : v_n = 0\}}{N} - \dfrac{1}{2}\right| < \delta$ and $\left|\dfrac{\#\{n : v_n = 1\}}{N} - \dfrac{1}{2}\right| < \delta$. Consider the natural bijection V between $\mathcal{L}(2, N)$ and the set $\mathcal{L}$ of sequences of length N of symbols from $\{(00), (01)\}$. Then for $(1 - \epsilon)2^N$ many $v \in \mathcal{L}(2, N)$,

$$\left|\frac{\#\{n : V(v)_n = (00)\}}{N} - \frac{1}{2}\right| < \delta \quad \text{and} \quad \left|\frac{\#\{n : V(v)_n = (01)\}}{N} - \frac{1}{2}\right| < \delta.$$

We can construe each length N sequence $V(v)$ from $\mathcal{L}$ as a length $2N$ binary sequence $V^*(v)$. Under this identification, $\eta_v = \sum_{j=1}^{2N} V^*(v)_j 2^{-j} = \sum_{j=1}^N v_j 4^{-j}$. For any $v \in \mathcal{L}(2, N)$,

$$\#\{m : V^*(v)_m = 0\} = 2\#\{n : V(v)_n = (00)\} + \#\{n : V(v)_n = (01)\}.$$

So, for $(1 - \epsilon)2^N$ many $v \in \mathcal{L}(2, N)$,

$$\#\{m : V^*(v)_m = 0\} \geqslant 2(1/2 - \delta)N + (1/2 - \delta)N = 3/2N - 3\delta N.$$

Thus, $\#\{m : \{2^m \eta_v\} \in [0, 1/2) \text{ and } 0 \leqslant m < 2N\} \geqslant (3/2)N - 3\delta N$. Hence,

$$\frac{1}{2N}\#\{m : \{2^m \eta_v\} \in [0, 1/2)\} \geqslant 3/4 - 3\delta/2.$$

For $\delta = 1/12$, the lemma follows. □

Lemma 3.14. *Let ϵ be a positive real and let s be a base. There is a k_0 such that for every $k \geqslant k_0$ there is an N_0 such that for all $N \geqslant N_0$,*

$$\#\Big\{w \in \mathcal{L}(\tilde{s}, N) : D(\{s^j \eta_w\} : 0 \leqslant j < kN) < \epsilon\Big\} > (1/2)\tilde{s}^N,$$

where $\tilde{s}$ is either of $s^k - 1$ or $s^k - 2$, and for $w = (w_1, \ldots, w_N) \in \mathcal{L}(\tilde{s}, N)$, $\eta_w = \sum_{j=1}^N w_j (s^k)^{-j}$. Furthermore, k_0 is a computable function of s and ϵ and N_0 is a computable function of s, ϵ and k.

Proof. Fix the real ϵ (to be used only at the end of the proof) and fix the base s. By Lemma 3.12, for each real $\delta > 0$ and integer $\ell > 0$ there is k_0 such that $\ell/k_0 < \delta$ and for all $k \geqslant k_0$

$$\#\Big\{v \in \mathcal{L}(s, k) : C(\ell, v) < \delta\Big\} > (1 - \delta)s^k.$$

Consider such a k. The elements $v \in \mathcal{L}(s, k)$ are of two types: those good-for-ℓ with $C(\ell, v) < \delta$ and the others. By choice of k, $(1 - \delta)s^k$ blocks of length k are good-for-ℓ. Let $\tilde{s}$ be either $s^k - 1$ or $s^k - 2$. Now view $\mathcal{L}(\tilde{s}, 1)$ in base s. If $\tilde{s}$ is $s^k - 1$, then $\mathcal{L}(\tilde{s}, 1)$ lacks the not-good-for-ℓ block of k digits all equal to $s - 1$. If $\tilde{s}$ is $s^k - 2$, then $\mathcal{L}(\tilde{s}, 1)$ also lacks the not-good-for-ℓ block of $k - 1$ digits equal to $s - 1$ followed by the final digit $s - 2$. So, at least $(1 - \delta)$ of the elements in $\mathcal{L}(\tilde{s}, 1)$ are good-for-ℓ in that they correspond to good blocks of length k. Let N_0 be such that for all $N \geqslant N_0$,

$$\#\Big\{w \in \mathcal{L}(\tilde{s},\ N) : C(1, w) < \delta\Big\} > (1 - \delta)\tilde{s}^N.$$

Take $N \geqslant N_0$ and consider a sequence w in $\mathcal{L}(\tilde{s}, N)$. If $C(1, w) < \delta$, then each element in $\mathcal{L}(\tilde{s}, 1)$ occurs in w at least $N(1/\tilde{s} - \delta)$ times. Let $w \mapsto w^*$ denote the map that takes $w \in \mathcal{L}(\tilde{s}, N)$ to $w^* \in \mathcal{L}(s, kN)$ such that $\displaystyle\sum_{n=1}^{N} w_n (s^k)^{-n} = \sum_{n=1}^{kN} w^*_{n+1}\, s^{-n}$. Let $u \in \mathcal{L}(s, \ell)$. We obtain the following bounds for $occ(u, w^*)$:

$$\begin{aligned}
occ(u, w^*) &\leqslant N(1/s^\ell + \delta)k + 2\ell N + \delta N k \\
&\leqslant Nk(1/s^\ell + 2\delta + 2\ell/k). \\
occ(u, w^*) &\geqslant \sum_{i=0}^{N-1} occ(u, (w^*_{ik+1}, \ldots, w^*_{ik+k})) \\
&\geqslant \tilde{s}(1 - \delta)N(1/\tilde{s} - \delta)k(1/s^\ell - \delta) = Nk(1 - \delta)(1 - \tilde{s}\delta)(1/s^\ell - \delta) \\
&\geqslant Nk(1/s^\ell - \delta - s^k\delta/s^\ell - \tilde{s}\delta^3) \\
&\geqslant Nk(1/s^\ell - \delta s^k). \quad \text{(We can assume that } \delta < 1/2.)
\end{aligned}$$

So $C(\ell, w^*) < \delta s^k$. Hence, $\#\Big\{w \in \mathcal{L}(\tilde{s}, N) : C(\ell, w^*) < \delta s^k\Big\} \geqslant (1-\delta)\tilde{s}^N$. Let $\delta = s^{-k}(\epsilon^2/18)$. Then,

$$\#\Big\{w \in \mathcal{L}(\tilde{s}, N) : C(\ell, w^*) < \epsilon^2/18\})\Big\} \geqslant (1 - s^{-k}(\epsilon^2/18))\tilde{s}^N.$$

In particular, this inequality holds for the minimal ℓ satisfying $s^\ell > 3/\epsilon$. Since, ϵ can be chosen so that $(1 - s^{-k}(\epsilon^2/18))$ is at least $1/2$, we can apply Lemma 3.11 to conclude the wanted result: $\#\Big\{w \in \mathcal{L}(\tilde{s}, N) : D(\{s^j \eta_w\} : 0 \leqslant j < kN) < \epsilon\Big\} > (1/2)\tilde{s}^N$. □

3.3 Schmidt's Lemmas

Lemma 3.17 is our analytic tool to control discrepancy for multiplicatively independent bases. It originates in Schmidt (1961/1962). Our proof adapts the version given in Pollington (1981).

Lemma 3.15 (Hilfssatz 5, Schmidt (1961/1962))**.** *Suppose that r and s are multiplicative independent bases. There is a constant c, with $0 < c < 1/2$, depending only on r and s, such that for all natural numbers K and ℓ with $\ell \geqslant s^K$,*

$$\sum_{r=0}^{N-1} \prod_{k=K+1}^{\infty} |\cos(\pi r^n \ell / s^k)| \leqslant 2N^{1-c}.$$

Furthermore, c is a computable function of r and s.†

Definition 3.16. $A(\xi, R, T, a, \ell) = \sum_{t \in T} \sum_{r \in R} \Big| \sum_{j=\langle a;r\rangle+1}^{\langle a+\ell;r\rangle} e(r^j t \xi)\Big|^2.$

Lemma 3.17. *Let R be a finite set of bases, T be a finite set of non-zero integers and a be a non-negative integer. Let s be a base multiplicatively independent to the elements of R and let $c(R, s)$ be the minimum of the constants c in Lemma 3.15 for pairs r, s with $r \in R$. Let $\tilde{s}$ be $s-1$ if s is odd and be $s-2$ if s is even. Let η be s-adic with precision $\langle a; s\rangle$. For $v \in \mathcal{L}(\tilde{s}, N)$ let η_v denote the rational number $\eta + s^{-\langle a;s\rangle} \sum_{j=1}^N v_j s^{-j}$. There is a length ℓ_0 such that for all $\ell \geqslant \ell_0$, there are at least $(1/2)\tilde{s}^{\langle a+\ell;s\rangle - \langle a;s\rangle}$ numbers η_v such that $A(\eta_v, R, T, a, \ell) \leqslant \ell^{\,2-c(R,s)/4}$. Furthermore, ℓ_0 is a computable function of R, T and s.*

Proof. We abbreviate $A(x, R, T, a, \ell)$ by $A(x)$, abbreviate $(a + \ell)$ by b and $\mathcal{L}(\tilde{s}, \langle b; s\rangle - \langle a; s\rangle)$ by $\mathcal{L}$. To provide the needed ℓ_0 we will estimate the mean value of $A(x)$ on the set of numbers η_v. We need an upper bound for

$$\sum_{v \in \mathcal{L}} A(\eta_v) = \sum_{v \in \mathcal{L}} \sum_{t \in T} \sum_{r \in R} \Big| \sum_{j=\langle a;r\rangle+1}^{\langle b;r\rangle} e(r^j t \eta_v)\Big|^2 = \sum_{v \in \mathcal{L}} \sum_{t \in T} \sum_{r \in R} \sum_{g=\langle a;r\rangle+1}^{\langle b;r\rangle} \sum_{j=\langle a;r\rangle+1}^{\langle b;r\rangle} e((r^j - r^g) t \eta_v).$$

Our main tool is Lemma 3.15, but it does not apply to all the terms $A(x)$ in the sum. So we will split it into two smaller sums over $B(x)$ and $C(x)$, so that a straightforward analysis

†Actually, Schmidt asserts the computability of c in separate paragraph (page 309 in the same article): "Wir stellen zunächst fest, daßman mit etwas mehr Mühe Konstanten $a_{20}(r, s)$ aus Hilfssatz 5 explizit berechnen könnte, und daß dann ξ eine eindeutig definierte Zahl ist."

applies to the first, and Lemma 3.15 applies to the other. Let p be the least integer satisfying the conditions for each $t \in T$, $r^{p-1} \geqslant 2|t|$ and for each $r \in R$, $r^p \geqslant s^2+1$.

$$B(x) = \sum_{t\in T}\sum_{r\in R}\left(\begin{array}{l} \displaystyle\sum_{g=\langle b;r\rangle-p+1}^{\langle b;r\rangle}\ \sum_{j=\langle a;r\rangle+1}^{\langle b;r\rangle} e((r^j-r^g)tx) \quad + \\ \displaystyle\sum_{g=\langle a;r\rangle+1}^{\langle b;r\rangle}\ \sum_{j=\langle b;r\rangle-p+1}^{\langle b;r\rangle} e((r^j-r^g)tx) \quad + \\ \displaystyle\sum_{g=\langle a;r\rangle+1}^{\langle b;r\rangle}\ \sum_{\substack{j=\langle a;r\rangle+1\\|g-j|<p}}^{\langle b;r\rangle} e((r^j-r^g)tx). \end{array}\right)$$

Assume for each $r \in R$, $\ell \geqslant \log r$ and $\ell \geqslant (8p\log s)^2$ (and recall, $b = a+\ell$.) We obtain the following bounds. The first inequality uses that each term in the explicit definition of $B(x)$ has norm less than or equal to 1. The second uses the assumed conditions on ℓ and the last inequality uses that $c(R,s) < 1/2$ as ensured by Lemma 3.15.

$$\begin{aligned}|B(x)| &\leqslant \sum_{t\in T}\sum_{r\in R} 4p(\langle b;r\rangle - \langle a;r\rangle) = \#T\,\#R\;4\,p\,(2\log s/\log r)(\langle b;s\rangle - \langle a;s\rangle)\\ &\leqslant \#T\,\#R\;(\langle b;s\rangle - \langle a;s\rangle)^{3/2}\\ &\leqslant \#T\,\#R\;(\langle b;s\rangle - \langle a;s\rangle)^{2-c(R,s)/2}.\end{aligned}$$

Thus, $\displaystyle\sum_{v\in\mathcal{L}} B(\eta_v) \leqslant \#T\,\#R\;(b-a)^{2-c(R,s)/2}\,\tilde{s}^{\langle b;s\rangle-\langle a;s\rangle}$. We estimate $\sum_{v\in\mathcal{L}} C(\eta_v)$, where

$$C(x) = \sum_{t\in T}\sum_{r\in R}\sum_{g=\langle a;r\rangle+1}^{\langle b;r\rangle-p}\ \sum_{\substack{j=\langle a;r\rangle+1\\|j-g|\geqslant p}}^{\langle b;r\rangle-p} e((r^j-r^g)tx).$$

We will rewrite $C(x)$ conveniently. We start by rewriting $\sum_{v\in\mathcal{L}} A(\eta_v)$.

$$\begin{aligned}\sum_{v\in\mathcal{L}} A(\eta_v) &= \sum_{t\in T}\ \sum_{r\in R}\ \sum_{v\in\mathcal{L}}\ \sum_{g=\langle a;r\rangle+1}^{\langle b;r\rangle}\ \sum_{j=\langle a;r\rangle+1}^{\langle b;r\rangle} e((r^j-r^g)t\eta_v)\\ &= \sum_{t\in T}\ \sum_{r\in R}\ \sum_{j=\langle a;r\rangle+1}^{\langle b;r\rangle}\ \sum_{g=\langle a;r\rangle+1}^{\langle b;r\rangle}\ \sum_{v\in\mathcal{L}} e((r^j-r^g)t\eta_v).\end{aligned}$$

For fixed j and g, we have the following identity.

$$\sum_{v\in\mathcal{L}} e((r^j-r^g)t\eta_v) = \prod_{k=\langle a;r\rangle+1}^{\langle b;r\rangle}\Big(1+e\Big(\frac{t(r^j-r^g)}{s^k}\Big)+\cdots+e\Big(\frac{(\tilde{s}-1)t(r^j-r^g)}{s^k}\Big)\Big).$$

Since $v \in \mathcal{L} = \mathcal{L}(\tilde{s}, \langle b;s\rangle - \langle a;s\rangle)$ the digits in v are in $\{0,\dots,\tilde{s}-1\}$. Thus,

$$\Big|\sum_{v\in\mathcal{L}} A(\eta_v)\Big| \leqslant \sum_{t\in T}\sum_{r\in R}\sum_{j=\langle a;r\rangle+1}^{\langle b;r\rangle}\ \sum_{g=\langle a;r\rangle+1}^{\langle b;r\rangle}\ \prod_{k=\langle a;r\rangle+1}^{\langle b;r\rangle}\Big|\sum_{d=0}^{\tilde{s}-1} e\Big(\frac{dt(r^j-r^g)}{s^k}\Big)\Big|$$

and

$$\Big|\sum_{v\in\mathcal{L}} C(\eta_v)\Big| \leqslant \sum_{t\in T}\sum_{r\in R}\sum_{j=\langle a;r\rangle+1}^{\langle b;r\rangle-p}\ \sum_{\substack{g=\langle a;r\rangle+1\\ |j-g|\geqslant p}}^{\langle b;r\rangle-p}\ \prod_{k=\langle a;r\rangle+1}^{\langle b;r\rangle}\Big|\sum_{d=0}^{\tilde{s}-1} e\Big(\frac{dt(r^j-r^g)}{s^k}\Big)\Big|.$$

Since $|\sum_x e(x)| = |\sum_x e(-x)|$, we can bound the sums over g and j as follows.

$$\Big|\sum_{v\in\mathcal{L}} C(\eta_v)\Big| \leqslant 2\sum_{t\in T}\sum_{r\in R}\sum_{j=p}^{\langle b;r\rangle-\langle a;r\rangle-p}\ \sum_{g=1}^{\langle b;r\rangle-\langle a;r\rangle-p-j}\ \prod_{k=\langle a;r\rangle+1}^{\langle b;r\rangle}\Big|\sum_{d=0}^{\tilde{s}-1} e\Big(\frac{dtr^{\langle a;r\rangle}r^g(r^j-1)}{s^k}\Big)\Big|.$$

Let $L=(r^j-1)r^{\langle a;r\rangle}t$. The following bounds related to L are ensured by the choice of p. Let $T_{\max}$ be the maximum of the absolute values of the elements of T.

$$\begin{aligned}
Lr^g s^{-\langle b;s\rangle} &\leqslant (r^j-1)r^{\langle a;r\rangle}tr^g s^{-\langle b;s\rangle}\\
&\leqslant r^j r^{\langle a;r\rangle}tr^{\langle b;r\rangle-\langle a;r\rangle-p-j}s^{-\langle b;s\rangle} = tr^{\langle b;r\rangle-p}s^{-\langle b;s\rangle}\\
&\leqslant T_{\max}\, r^{\lceil b/\log r\rceil}\ s^{-\lceil b/\log s\rceil}r^{-p}\\
&\leqslant T_{\max}\, r^{1-p}\\
&\leqslant 1/2 \qquad \text{(an ensured condition on } p\text{)}.
\end{aligned}$$

We give a lower bound on the absolute value of L.

$$\begin{aligned}
|L| &\geqslant (r^p-1)r^{\langle a;r\rangle} = (r^p-1)r^{\lceil a/\log r\rceil}\\
&\geqslant (r^p-1)s^{a/\log s}\\
&\geqslant s^{2+a/\log s} \qquad \text{(an ensured condition on } p\text{)}\\
&\geqslant s^{\langle a;s\rangle+1}.
\end{aligned}$$

Below, we use $\Big|\sum_{d=0}^{\tilde{s}-1} e(dx)\Big| \leqslant (\tilde{s}/2)\ |1+e(x)|$; notice that the leading coefficient is whole (note to the curious reader: this the only reason that $\tilde{s}$ is required to be even).

$$\sum_{g=1}^{\langle b;r\rangle-\langle a;r\rangle-p-j}\ \prod_{k=\langle a;r\rangle+1}^{\langle b;r\rangle}\Big|\sum_{d=0}^{\tilde{s}-1} e(dLr^g s^{-k})\Big| \leqslant \sum_{g=1}^{\langle b;r\rangle-\langle a;r\rangle-p-j}\ \prod_{k=\langle a;r\rangle+1}^{\langle b;r\rangle}\frac{\tilde{s}}{2}\Big|1+e\Big(r^g L s^{-k}\Big)\Big|$$

which, by the double angle identities, is at most $\tilde{s}^{\langle b;s\rangle-\langle a;s\rangle}\displaystyle\sum_{g=1}^{\langle b;r\rangle-\langle a;r\rangle-p-j}\ \prod_{k=\langle a;r\rangle+1}^{\langle b;r\rangle}|\cos(\pi L r^g s^{-k})|$.

If $k\geqslant\langle b;r\rangle$, then $Lr^g s^{-k}\leqslant 2^{-(k+1)}$. Therefore, $\displaystyle\prod_{k=\langle b;r\rangle+1}^{\infty}|\cos(\pi Lr^g s^{-k})| \geqslant \prod_{k=1}^{\infty}|\cos(\pi 2^{-(k+1)})|$, where the right hand side is a positive constant. Then,

$$\prod_{k=\langle a;r\rangle+1}^{\langle b;r\rangle}|\cos(\pi Lr^g s^{-k})| = \prod_{k=\langle a;r\rangle+1}^{\infty}|\cos(\pi Lr^g s^{-k})|\ \left(\prod_{k=\langle b;r\rangle+1}^{\infty}|\cos(\pi Lr^g s^{-k})|\right)^{-1}$$

which, for the appropriate constant $\tilde{c}$, is at most $\tilde{c} \prod_{k=\langle a;r\rangle+1}^{\infty} |\cos(\pi L r^g s^{-k})|$.

We can apply Lemma 3.15:

$$\sum_{g=1}^{\langle b;r\rangle-\langle a;r\rangle-p-j} \prod_{k=\langle a;r\rangle+1}^{\langle b;r\rangle} \Big| \sum_{d=0}^{\tilde{s}-1} e(dLr^g s^{-k}) \Big| \leqslant \sum_{g=1}^{\langle b;r\rangle-\langle a;r\rangle-p-j} \tilde{c} \prod_{k=\langle a;r\rangle+1}^{\infty} |\cos(\pi L r^g s^{-k})|$$
$$\leqslant 2\tilde{c}(\langle b;r\rangle - \langle a;r\rangle)^{1-c(R,s)}.$$

$$\Big|\sum_{v\in\mathcal{L}} C(\eta_v)\Big| \leqslant 2\sum_{t\in T}\sum_{r\in R}\sum_{j=p}^{\langle b;r\rangle-\langle a;r\rangle-p} \tilde{s}^{\langle b;s\rangle-\langle a;s\rangle}\, 2\tilde{c}(\langle b;s\rangle - \langle a;s\rangle)^{1-c(R,s)}$$
$$\leqslant 4\tilde{c}\,\#T\,\#R\,(\langle b;s\rangle - \langle a;s\rangle)^{2-c(R,s)}\,\tilde{s}^{\langle b;s\rangle-\langle a;s\rangle}.$$

Combining this with the estimate for $|\sum_{v\in\mathcal{L}} B(\eta_v)|$, we have

$$|\sum_{v\in\mathcal{L}} A(\eta_v)| \leqslant 4\tilde{c}\,\#T\#R(\langle b;s\rangle - \langle a;s\rangle)^{2-c(R,s)}\tilde{s}^{\langle b;s\rangle-\langle a;s\rangle}.$$

Therefore, the number of $v \in \mathcal{L}$ such that $A(\eta_v) > 4\tilde{c}\,\#T\,\#R(\langle b;s\rangle - \langle a;s\rangle)^{-c(R,s)/2}$ is at most $(\langle b;s\rangle - \langle a;s\rangle)^{-c(R,s)/2}\,\tilde{s}^{\langle b;s\rangle-\langle a;s\rangle}$. If $\ell > (2^{2/c(R,s)}+1)\log s$ and $\ell > (16\tilde{c}\#T\#R)^{4/c(R,s)}$ then $(\langle b;s\rangle - \langle a;s\rangle)^{-c(R,s)/2} < 1/2$. So, there are at least $(1/2)\tilde{s}^{(\langle b;s\rangle-\langle a;s\rangle)}$ members $v \in \mathcal{L}$ for which

$$\begin{aligned} A(\eta_v) &\leqslant 4\tilde{c}\,\#T\,\#R(\langle b;s\rangle - \langle a;s\rangle)^{2-c(R,s)/2} \\ &\leqslant 4\tilde{c}\,\#T\,\#R(2\ell)^{2-c(R,s)/2} \\ &\leqslant \ell^{\,2-c(R,s)/4}. \end{aligned}$$

This proves the lemma for ℓ_0 equal to the least integer greater than $(2^{2/c(R,s)}+1)\log s$, $(16\tilde{c}\#T\#R)^{4/c(R,s)}$, $(8p\log s)^2$ and $\max\{\log r : r \in R\}$. □

3.4 On Changing Bases

Lemma 3.18. *For any interval I and base s, there is a s-adic subinterval I_s such that $\mu(I_s) \geqslant \mu(I)/(2s)$.*

Proof. Let m be least such that $1/s^m < \mu(I)$. Note that $1/s^m \geqslant \mu(I)/s$, since $1/s^{m-1} \geqslant \mu(I)$. If there is a s-adic interval of length $1/s^m$ strictly contained in I, then let I_s be such an interval, and note that I_s has length greater than or equal to $\mu(I)/s$. Otherwise, there must be an a such that a/s^m is in I and neither $(a-1)/s^m$ nor $(a+1)/s^m$ belongs to I. Thus, $2/s^m > \mu(I)$. However, since $1/s^m < \mu(I)$ and $s \geqslant 2$ then $2/s^{m+1} < \mu(I)$. So, at least one of the two intervals $\left[\frac{sa-1}{s^{m+1}}, \frac{sa}{s^{m+1}}\right)$ or $\left[\frac{sa}{s^{m+1}}, \frac{sa+1}{s^{m+1}}\right)$ must be contained in I. Let I_s be such. Then, $\mu(I_s)$ is $\frac{1}{s^{m+1}} = \frac{1}{2s}\frac{2}{s^m} > \mu(I)/(2s)$. In either case, the length of I_s is greater than $\mu(I)/(2s)$. □

Lemma 3.19. *Let s_0 and s_1 be bases and suppose that I is an s_0-adic interval of length $s_0^{-\langle b;s_0\rangle}$. For $a = b + \lceil \log s_0 + 3\log s_1 \rceil$, there is an s_1-adic subinterval of I of length $s_1^{-\langle a;s_1\rangle}$.*

Proof. By the proof of Lemma 3.18, there is an s_1-adic subinterval of I of length $s_1^{-(\lceil -\log_{s_1}(\mu(I))\rceil+1)}$:

$$\begin{aligned}\lceil -\log_{s_1}(\mu(I))\rceil + 1 = \lceil -\log_{s_1}(s_0^{-\langle b;s_0\rangle})\rceil + 1 &= \lceil \langle b;s_0\rangle \log s_0/\log s_1\rceil + 1\\ &\leqslant \lceil b/\log s_1 + \log s_0/\log s_1\rceil + 1\\ &\leqslant \langle b;s_1\rangle + \lceil \log s_0/\log s_1\rceil + 1.\end{aligned}$$

Thus, there is an s_1-adic subinterval of I of length $s_1^{-(\langle b;s_1\rangle+\lceil \log s_0/\log s_1\rceil+1)}$. Consider $a = b + \lceil \log s_0 + 3\log s_1\rceil$. Then

$$\begin{aligned}\langle a;s_1\rangle = \lceil a/\log s_1\rceil &= \lceil b + \lceil \log s_0 + 3\log s_1\rceil/\log s_1\rceil\\ &\geqslant b/\log s_1 + (\log s_0 + 3\log s_1)/\log s_1\\ &\geqslant \langle b;s_1\rangle + \lceil \log s_0/\log s_1\rceil + 1.\end{aligned}$$

This inequality is sufficient to prove the lemma. □

The next observation is by direct substitution. We will use it in the proofs of the theorems.

Remark 3.20. Suppose that r, s_0 and s_1 are bases. Let b be a positive integer and let $a = b + \lceil \log s_0 + 3\log s_1\rceil$. Then, $\langle a;r\rangle - \langle b;r\rangle \leqslant \lceil \log s_0 + 3\log s_1\rceil/\log r + 1$. Hence, $\langle a;r\rangle - \langle b;r\rangle \leqslant 2\lceil \log s_0 + 3\log s_1\rceil$.

4 Proofs of Theorems

4.1 Tools

Notation. Let M be the set of minimal representatives of the multiplicative dependence equivalence classes. Let $p(s_0, s_1) = 2\lceil \log s_0 + 3\log s_1\rceil$.

Definition 4.1. Let T and δ be as defined in Lemma 3.8 for input $(\epsilon/10)^4$. Let ℓ be the function with inputs R, s, k, ϵ and value the least integer greater than all of the following:

- The maximum of ℓ_0 as defined in Lemma 3.9 over all inputs r in R and ϵ as given.
- N_0 as defined in Lemma 3.14 for inputs s, k and ϵ
- ℓ_0 as defined in Lemma 3.17 for inputs R, T and s^k.
- $\big((\log r)^2/\delta\big)^{4/c(R,s^k)}$ for $c(R, s^k)$ the minimum of the constants of Lemma 3.15 for pairs s, r with $r \in R$.

4.2 Proof of Theorem 2

Theorem 2. *For any Π^0_3 subset R of M there is a computable real number ξ such that for all $r \in M$, $r \in R$ if and only if ξ is normal to base r. Furthermore, ξ is computable uniformly in the Π^0_3 formula that defines R.*

Note that $m \in M$ if and only if there is no n less than m such that m is an integer power of n, an arithmetic condition expressed using only bounded quantification. Let $\varphi = \forall x\exists y\forall z\theta$ be a Π^0_3 formula with one free variable. We will construct a real number ξ so that for every base r, ξ is normal to base r if and only if $\varphi(r)$ is true. The normality of ξ to base r is naturally expressed using three quantifiers: $\forall\epsilon\exists n\forall N \geqslant n\ D(\{r^k\xi\} : 0 \leqslant k < N) < \epsilon$. Lemma 3.1 shows that the discrepancy D admits computable approximations (using finite partitions of the unit interval). Thus, the normality of ξ to base r is a Π^0_3 formula. In our construction, we will bind the quantified variables in $\varphi(r)$ to those in the formula for normality. The variable x will correspond to ϵ, y to n and z to N.

We define a sequence ξ_m, b_m, s_m, k_m, ϵ_m, ℓ_m, x_m, R_m and c_m by stages. ξ_m is a $s_m^{k_m}$-adic rational number of precision $\langle b_m; s_m^{k_m}\rangle$. b_m and k_m are positive integers. s_m is a base. R_m is a finite set of bases. The real ξ will be an element of $[\xi_m, \xi_m + (s_m^{k_m})^{-\langle b_m; s_m^{k_m}\rangle})$. Stage $m+1$ is devoted to extending ξ_m so that the discrepancy of the extended part in base s_{m+1} is below $1/x_m$ and above $1/(2s_{m+1}^{k_{m+1}})$, and so that the discrepancy of the extension for the other bases under consideration is below ϵ_{m+1}. ℓ_{m+1} is used to determine the length of the extension and c_{m+1} is an integer used to monitor φ and set bounds on discrepancy. Fix an enumeration of M such that every element of M appears infinitely often.

Initial stage. Let $\xi_0 = 0$, $b_0 = 1$, $s_0 = 3$, $k_0 = 1$, $\epsilon_0 = 1$, $\ell_0 = 1$, $x_0 = 1$ and $c_0 = 1$.

Stage $m+1$. Given ξ_m of the form $\sum_{j=1}^{\langle b_m; s_m^{k_m}\rangle} v_j (s_m^{k_m})^{-j}$, b_m, s_m, k_m, ϵ_m, ℓ_m, x_m, R_m and c_m.

(1) Let F be the canonical partition of $[0,1]$ into intervals of length $(1/3)(1/(4)s_m^{-k_m}$. If $D(F, (\{s_m^j\xi_m\} : 0 \leqslant j < \langle b_m; s_m\rangle)) < ((1/3)(1/4)s_m^{-k_m}))^2$, then let s_{m+1} be s_m, k_{m+1} be k_m, ϵ_{m+1} be ϵ_m, ℓ_{m+1} be ℓ_m, x_{m+1} be x_m, R_{m+1} be R_m and c_{m+1} be c_m.

(2) Otherwise, let c be $c_m + 1$. Let s be the cth element in the enumeration of M. Let n be maximal less than c such that s is also the nth element in the enumeration of M, or be 0 if s appears for the first time at c. Take x to be minimal such that there is a y less than n satisfying $\forall z < n\, \varphi(x, y, z)$ and $\exists z < c\, \neg\varphi(x, y, z)$. If there is none such, then set x equal to c. Let k and N be as defined in Lemma 3.14 for input $\epsilon = 1/x$ and base s. Let R be the set of bases not equal to s which appear in the enumeration of M at positions less than c. Let L be the least integer greater than $\max\{x, c, 2s^k\}\log(\max(R \cup \{s\}))p(s_m, s)$, N, and $\ell(R, s, k, 1/c)$. If for some $r \in R$, $(1/c)\langle b_m; r\rangle \leqslant L + p(s_m, s)$ or $(1/x)\langle b_m; s\rangle \leqslant L + p(s_m, s)$ then let s_{m+1} be s_m, k_{m+1} be k_m, ϵ_{m+1} be ϵ_m, ℓ_{m+1} be ℓ_m, x_{m+1} be x_m, R_{m+1} be R_m and c_{m+1} be c_m.

(3) Otherwise, let s_{m+1} be s, k_{m+1} be k, ϵ_{m+1} be $1/c$, ℓ_{m+1} be L x_m be x, R_{m+1} be R and c_{m+1} be c.

Let a_{m+1} be minimal such that there is an $s_{m+1}^{k_{m+1}}$-adic subinterval of $[\xi_m, \xi_m + (s_m^{k_m})^{-\langle b_m; s_m^{k_m}\rangle})$ of length $(s_{m+1}^{k_{m+1}})^{-\langle a_{m+1}; s_{m+1}^{k_{m+1}}\rangle}$ and let $[\eta_{m+1}, \eta_{m+1} + (s_{m+1}^{k_{m+1}})^{-\langle a_{m+1}; s_{m+1}^{k_{m+1}}\rangle})$ be the leftmost such. Let $\tilde{s}$ be $s_{m+1}^{k_{m+1}} - 1$ if s_{m+1} is odd and be $s_{m+1}^{k_{m+1}} - 2$ otherwise. Let T and δ be as defined in Lemma 3.8 for input $\epsilon = (\epsilon_{m+1}/10)^4$. Let b_{m+1} be $a_{m+1} + \ell_{m+1}$. Let ν be such that

- $\nu = \eta_{m+1} + \displaystyle\sum_{j=\langle a_{m+1}; s_{m+1}^{k_{m+1}}\rangle+1}^{\langle b_{m+1}; s_{m+1}^{k_{m+1}}\rangle} w_j (s_{m+1}^{k_{m+1}})^{-j}$, for some $(w_1, \ldots, w_{\ell_{m+1}})$ in $\mathcal{L}(\tilde{s}, \ell_{m+1})$.
- $A(\nu, R_{m+1}, T, a_{m+1}, \ell_{m+1})/\langle \ell_{m+1}; \max(R_{m+1})\rangle^2 < \delta$.
- ν minimizes $D(F, (\{s_{m+1}^j \nu\} : \langle a_{m+1}; s_{m+1}\rangle < j \leqslant \langle b_{m+1}; s_{m+1}\rangle))$ among the ν satisfying the first two conditions, for F as defined in clause (1). If there is more than one minimizer, take the least such for ν.

We define ξ_{m+1} to be ν. This ends the description of stage $m+1$.

We verify that the construction succeeds. Let $m+1$ be a stage. If clause (1) or (2) applies, let m_0 be the greatest stage less than or equal to $m+1$ such that $c_{m_0} = c_{m_0+1} = \cdots = c_{m+1}$. During stage m_0, k_{m_0} and ℓ_{m_0} were chosen to satisfy the conditions to reach clause (3). Note that since $b_m > b_{m_0}$ the conditions in clause (2) apply to b_m in place of b_{m_0}:

$(1/c_{m+1})b_m > \ell_{m+1} + p(s_{m_0-1}, s_{m+1})$ and $(1/x_{m+1})b_m > \ell_{m+1} + p(s_{m_0-1}, s_{m+1})$. Then, ℓ_{m+1} is greater than $\max\{x_{m+1}, c_{m+1}, 2s_{m+1}^{k_{m+1}}\} \log(\max(R_{m+1} \cup \{s_{m+1}\})p(s_{m_0-1}, s_{m+1})$, N, and $\ell(R_{m+1}, s_{m+1}, k_{m+1}, 1/c_{m+1})$, where N is determined during stage m_0. If clause (3) applies, then the analogous conditions hold by construction.

Stage $m+1$ determines the $s_{m+1}^{k_{m+1}}$-adic subinterval $[\eta_{m+1}, \eta_{m+1} + (s_{m+1}^{k_{m+1}})^{-\langle a_{m+1}; s_{m+1}^{k_{m+1}}\rangle})$ of the interval provided at the end of stage m. The existence of this subinterval is ensured by Lemma 3.19. The stage ends by selecting the rational number ν. The existence of an appropriate ν is ensured by Lemma 3.17 with the inputs given by the construction. It follows that ξ is well defined as the limit of the ξ_m.

Let s be a base that appears in the enumeration of M at or before c_{m+1}. There are two possibilities for s during stage m: either it is an element of R_{m+1} or it is equal to s_{m+1}. Suppose first that $s \in R_{m+1}$. $\xi_{m+1} = \nu$ was chosen so that $A(\nu, R_{m+1}, T, a_{m+1}, \ell_{m+1})/\langle \ell_{m+1}; s\rangle^2 < \delta$. By Definition 3.16, $A(\nu, R_{m+1}, T, a_{m+1}, \ell_{m+1})$ is equal to $\sum_{t \in T} \sum_{r \in R_{m+1}} \Big| \sum_{j=\langle a_{m+1}+1; r\rangle}^{\langle b_{m+1}; r\rangle} e(r^j t\nu)\Big|^2$.

Hence, $(1/\langle \ell_{m+1}; s\rangle^2) \sum_{t \in T} \Big| \sum_{j=\langle a_{m+1}+1; s\rangle}^{\langle b_{m+1}; s\rangle} e(s^j t\nu)\Big|^2 < \delta$. By choice of T and δ, Lemma 3.8 ensures

$$D(s^j \nu : \langle a_{m+1}; s\rangle < j \leqslant \langle b_{m+1}; s\rangle) < (\epsilon_{m+1}/10)^4.$$

By definition of ξ, $\xi \in [\nu, \nu + (s_{m+1}^{k_{m+1}})^{-\langle b_{m+1}; s_{m+1}^{k_{m+1}}\rangle})$. By Lemma 3.9, we conclude that

$$D(s^j \xi : \langle a_{m+1}; s\rangle < j \leqslant \langle b_{m+1}; s\rangle) < \epsilon_{m+1}.$$

By Remark 3.20, $\langle a_{m+1}; s\rangle - \langle b_m; s\rangle$ is less than or equal to $p(s_m, s_{m+1})$. By construction, $\epsilon_{m+1}\langle \ell_{m+1}; s\rangle$ is greater than $p(s_m, s_{m+1})$. By Lemma 3.3

$$D(s^j \xi : \langle b_m; s\rangle < j \leqslant \langle b_{m+1}; s\rangle) < 2\epsilon_{m+1}.$$

The second possibility is that s is equal to s_{m+1}. Again, consider the selection of the rational number ν during stage $m+1$. By Lemma 3.17, more than half of the eligible candidates satisfy the inequality $A(\nu, R_{m+1}, T, b_m, \ell_{m+1})/\langle \ell_{m+1}; \max(R_{m+1})\rangle^2 < \delta$. By Lemma 3.14, more than half the candidates satisfy

$$D(\{s^j \nu\} : \langle a_{m+1}; s\rangle < j \leqslant \langle b_{m+1}; s\rangle) < 1/x_{m+1}.$$

By choice of ξ_{m+1},

$$D(F, (\{s^j \xi_{m+1}\} : \langle a_{m+1}; s\rangle < j \leqslant \langle b_{m+1}; s\rangle) < 1/x_{m+1}.$$

By Lemma 3.1,

$$D(\{s^j \xi_{m+1}\} : \langle a_{m+1}; s\rangle < j \leqslant \langle b_{m+1}; s\rangle) < 3x_{m+1}^{-1/2}.$$

By construction $(1/x_{m+1})\langle \ell_{m+1}; s\rangle$ is greater than $p(s_m, s)$ and so, as above,

$$D(\{s^j\xi_m\} : \langle b_m; s\rangle < j \leqslant \langle b_{m+1}; s\rangle) < 6x_{m+1}^{-1/2}.$$

By Lemma 3.9,

$$D(\{s^j\xi\} : \langle b_m; s\rangle < j \leqslant \langle b_{m+1}; s\rangle) < 10(6x_{m+1}^{-1/2})^{1/4}.$$

Further, ξ_{m+1} is chosen so that in $(\{(s^{k_{m+1}})^j\xi_{m+1}\} : \langle a_{m+1}; s^{k_{m+1}}\rangle < j \leqslant \langle b_{m+1}; s^{k_{m+1}}\rangle)$ no element belongs to $[1 - s^{k_{m+1}}, 1]$. As $\xi \in [\xi_{m+1}, \xi_{m+1} + s^{-\langle b_{m+1};s\rangle})$, the same holds for ξ. Since $\langle b_{m+1}; s^{k_{m+1}}\rangle - \langle b_m; s^{k_{m+1}}\rangle > 2s^{k_{m+1}}p(s_m, s)$, Lemma 3.5 applies and so

$$D(\{s^j\xi\} : \langle b_m; s\rangle < j \leqslant \langle b_{m+1}; s\rangle)) \geqslant 1/(2s^{k_{m+1}}).$$

Stages subsequent to $m + 1$ will satisfy the same inequality until the first stage for which $D(\{s_m^j\xi_m\} : 0 \leqslant j < \langle b_m; s_m\rangle) \geqslant 1/(4s_m^{k_m})$. By a direct counting argument, there will be such a stage and during that stage clause (1) cannot apply. Similarly, clause (2) cannot apply for indefinitely many stages, as the values of b_m are unbounded. It follows that $\lim_{m\to\infty} c_{m+1} = \infty$.

If $\varphi(s)$ is true, then for each x, there are only finitely many stages during which $s_m = s$ and $x_m = x$. Let ϵ be greater than 0. There will be a stage m_0 such that for all m greater than m, $\epsilon > 2\epsilon_m$ and, if $s = s_m$ then $\epsilon > 10(6x_{m+1}^{-1/2})^{1/4}$. By construction, Lemma 3.4 applies to conclude $\lim_{N\to\infty} D(\{s^j\xi_m\} : 0 \leqslant j < N) \leqslant 2\epsilon$. By applying Lemma 3.9, we conclude that ξ is normal to base s.

If $\varphi(s)$ is not true, then let x be minimal such that $\forall y\exists z\neg\varphi(s, x, y, z)$. There will be infinitely many $m + 1$ such that $s = s_{m+1}$, $x = x_{m+1}$ and k_{m+1} is the k associated with s and x. As already discussed, each of these stages will be followed by a later stage m_1 such that

$$D(\{s^j\xi\} : 0 \leqslant j < \langle b_{m_1}; s\rangle) \geqslant 1/(4s^k).$$

Hence ξ is not normal to base s.

4.3 Proof of Theorem 1

Theorem 1. *(1) The set of indices for computable real numbers which are normal to at least one base is Σ_4^0-complete. (2) The set of real numbers that are normal to at least one base is $\boldsymbol{\Sigma}_4^0$-complete.*

To prove item (1) we must exhibit a computable function f, taking Σ_4^0 sentences (no free variables) to indices for computable real numbers, such that for any Σ_4^0 sentence ψ, ψ is true in the natural numbers if and only if the computable real number named by $f(\psi)$ is normal to at least one base. Let ψ be a Σ_4^0 sentence and let φ be the Π_3^0 formula such that $\psi = \exists w\varphi(w)$. Let M be the set of minimal representatives of the multiplicative dependence equivalence classes and fix the computable enumeration of $M = \{s_1, s_2, \dots\}$ (as in the proof of Theorem 2). Consider the Π_3^0 formula φ^* such that $\varphi^*(s_w)$ is equivalent to $\varphi(w)$. By Theorem 2, there is a computable real ξ such that for all s_w, ξ is normal to base s_w if and only if $\varphi^*(s_w)$ is true, if and only if $\varphi(w)$ is true. Thus, ξ is normal to at least one base if and only if there is a w such that $\varphi(w)$ is true, if and only if $\psi = \exists w\varphi(w)$ is true. In Theorem 2, ξ is obtained uniformly from φ^*, which was obtained uniformly from φ. The result follows.

For item (2) recall that a subset in $\mathbb{R}$ is $\boldsymbol{\Sigma}_4^0$-complete if it is $\boldsymbol{\Sigma}_4^0$ and it is hard for $\boldsymbol{\Sigma}_4^0$. To prove hardness of subsets of $\mathbb{R}$ at levels in the Borel hierarchy it is sufficient to consider subsets of Baire space, $\mathbb{N}^\mathbb{N}$ because there is a continuous function from $\mathbb{R}$ to $\mathbb{N}^\mathbb{N}$ that preserves the levels. The Baire space admits a syntactic representation of the levels of Borel hierarchy in arithmetical terms. A subset A of $\mathbb{N}^\mathbb{N}$ is $\mathbf{\Sigma}^0_4$ if and only if there is a parameter p in $\mathbb{N}^\mathbb{N}$ and a Σ^p_4 formula $\psi(x,p)$, where x is a free variable, such that for all $x \in \mathbb{N}^\mathbb{N}$, $x \in A$ if and only if $\psi(x,p)$ is true. A subset B of $\mathbb{R}$ is hard for $\mathbf{\Sigma}^0_4$ if for every $\mathbf{\Sigma}^0_4$ subset A of $\mathbb{N}^\mathbb{N}$ there is a continuous function f such that for all $x \in \mathbb{N}^\mathbb{N}$, $x \in A$ if and only if $f(x) \in B$. Consider a $\mathbf{\Sigma}^0_4$ subset A of the Baire space defined by a Σ^p_4 formula $\psi(x,p)$, where x is a free variable. The same function given for item (1) but now relativized to x and p yields a real number ξ such that $\psi(x,p)$ is true if and only if ξ is normal to at least one base. This gives the required continuous function f satisfying $x \in A$ if and only $f(x)$ is normal to at least one base.

4.4 Proof of Theorem 3

Theorem 3. *For any Π^0_3 formula φ there is a computable real number ξ such that for any base $r \in M$, $\varphi(\xi, r)$ is true if and only if ξ is normal to base r.*

The proof follows from Theorem 2 by an application of the Kleene Fixed Point Theorem (see Rogers, 1987, Chapter 11). Let φ be a Π^0_3 formula with two free variables, one ranging over $\mathbb{N}^\mathbb{N}$ and the other ranging over $\mathbb{N}$. Let Ψ_e be a computable enumeration of the partial computable functions from $\mathbb{N}$ to $\mathbb{N}$. The condition "Ψ_e is a total function and $\varphi(\Psi_e, r)$" is a Π^0_3 property of e and r. By Theorem 2, there is a computable function which on input a Π^0_3 formula θ produces a (total) computation of a real ξ_θ which is normal to base $r \in M$ if and only if $\theta(r)$ is true. In particular, there is a computable function f such that for every e, for all $r \in M$,

Ψ_e is a total function and $\varphi(\Psi_e, r)$ if and only if $\Psi_{f(e)}$is normal to base r.

Furthermore, for every e, $\Psi_{f(e)}$ is total. By the Kleene Fixed Point Theorem, there is an e such that Ψ_e is equal to $\Psi_{f(e)}$. For this e, for all $r \in M$,

$\varphi(\Psi_e, r)$ if and only if Ψ_e is normal to base r.

Then, $\xi = \Psi_e$ satisfies the condition of the Theorem.

4.5 Proof of Theorem 4

Theorem 4. *Fix a base s. There is a computable function $f : \mathbb{N} \to \mathbb{Q}$ monotonically decreasing to 0 such that for any function $g : \mathbb{N} \to \mathbb{Q}$ monotonically decreasing to 0 there is an absolutely normal real number ξ whose discrepancy for base s eventually dominates g and whose discrepancy for each base multiplicatively independent to s is eventually dominated by f. Furthermore, ξ is computable from g.*

Let s be a base. We define a sequence ξ_m, b_m, k_m, ϵ_m, ℓ_m, R_m and $\overline{k}_m$ by stages. b_m, k_m and $\overline{k}_m$ are a positive integers, ϵ_m a positive rational number and R_m a finite set of bases multiplicatively independent to s. ξ_m is an s^{k_m}-adic rational number of precision $\langle b_m; s^{k_m}\rangle$. The real ξ will be an element of $[\xi_m, \xi_m + (s^{k_m})^{-\langle b_m; s^{k_m}\rangle})$. Stage $m+1$ is devoted to extending ξ_m so that the discrepancy of the extension is below ϵ_{m+1} for the bases in R_{m+1} and so that the discrepancy of the extension in base s is in a controlled interval above g. We use k_{m+1} to enforce the endpoints of this interval. ℓ_m determines the length of the extension.

At each stage m the determination of ξ_{m+1} is done so that the discrepancy functions for ξ relative to bases independent to s converge to 0 uniformly, without reference to the function g.

We obtain f as the function bounding these discrepancies by virtue of construction. The variable $\overline{k}_m$ acts as a worse case surrogate for the exponent of s used in the construction relative to g.

Initial stage. Let $\xi_0 = 0$, $b_0 = 1$, $k_0 = 1$, $\epsilon_0 = 1$, $\ell_0 = 0$, $R_0 = \{r_0\}$ where r_0 is the least base which is multiplicatively independent to s, and $\overline{k}_0 = 1$.

Stage $m+1$. Given b_m, R_m, ϵ_m, $\overline{k}_m$, k_m and ξ_m of the form $\sum_{j=1}^{\langle b_m; s^{k_m}\rangle} v_j(s^{k_m})^{-j}$.

(1) Let r be the least number greater than the maximum element of R_m which is multiplicatively independent to s. If $(\epsilon_m/2)\langle b_m; r\rangle \geqslant \ell(R_m \cup \{r\}, s, \overline{k}_m + 1, \epsilon_m/2)$ then let ϵ_{m+1} be $\epsilon_m/2$, let R_{m+1} be $R_m \cup \{r\}$ and $\overline{k}_{m+1}$ be $\overline{k}_m + 1$. Otherwise, let ϵ_{m+1} be ϵ_m, R_{m+1} be R_m and $\overline{k}_{m+1}$ be $\overline{k}_m$. Let $\ell_{m+1} = \ell(R_{m+1}, s, \overline{k}_{m+1}, \epsilon_{m+1})$ and let $b_{m+1} = b_m + \ell_{m+1}$.

(2) Let k and N be as determined by Lemma 3.14 for the input value $\epsilon = 1/(4s^{k_m})$. If $(k \leqslant \overline{k}_{m+1})$, $(N \leqslant \langle \ell_{m+1}; s\rangle)$ and $(1/(2s^k) > g(\langle b_m; s\rangle))$, then let k_{m+1} be k. Otherwise, let k_{m+1} be k_m.

We define ξ_{m+1} to be $\xi_m + \nu$, where ν is determined as follows. Let $\tilde{s}$ be $s^{k_{m+1}} - 1$ if s is odd and be $s^{k_{m+1}} - 2$ if s is even. Let T and δ be as determined in Lemma 3.8 with input $\epsilon = (\epsilon_{m+1}/10)^4$. Let ν be such that

- $\nu = \xi_m + \displaystyle\sum_{j=1}^{\langle \ell_{m+1}; s^{k_{m+1}}\rangle} w_j(s^{k_{m+1}})^{-(\langle b_m; s^{k_{m+1}}\rangle + j)}$ for some $(w_1, \dots, w_{\langle \ell_{m+1}; s_{m+1}^{k_{m+1}}\rangle})$ in $\mathcal{L}(\tilde{s}, \langle \ell_{m+1}; s_{m+1}^{k_{m+1}}\rangle)$.
- $A(\nu, R_{m+1}, T, \ell_{m+1})/\langle \ell_{m+1}; \max(R_{m+1})\rangle^2 < \delta$.
- ν minimizes $D(F, (\{s^j\nu\} : 0 \leqslant j < \langle \ell_{m+1}; s\rangle))$ among the ν satisfying the first two conditions, where F is the canonical partition of $[0,1]$ into intervals of length $(1/3)(1/4s^{k_{m+1}})$. If there is more than one minimizer, take the least such for ν.

We define the function $f : \mathbb{N} \to \mathbb{Q}$ as follows. Given a positive integer n, let m_n be such that $b_{m_n} \leqslant n < b_{m_n+1}$. Let m_0 be maximal such that $\epsilon_{m_0}\langle b_{m_n}; \max(R_{m_0})\rangle > b_{m_0}$. Define $f(n)$ to be $4\epsilon_{m_0}$. By construction, ϵ_m is monotonically decreasing and so f is also. Note, for all m, $\ell_m > 0$ and $\lim_{m\to\infty} b_m = \infty$. For every stage $m+1$, clause (1) sets ϵ_{m+1} to be $\epsilon_m/2$, unless b_m is not sufficiently large. The value of ϵ_{m+1} will be reduced at a later sufficiently large stage. Thus, ϵ_m goes to 0 and so does f.

The function f is defined in terms of the sequences of values b_m, R_m and ϵ_m, which are determined by clause (1). The conditions and functions appearing in clause (1) are computable, as was verified in each of the relevant lemmas. Thus, f is a computable function.

Suppose that r and s are multiplicatively independent. Fix n_0 and n_1 so that $r \in R_{n_0}$ and $\epsilon_{n_0}\langle b_{n_1}; \max(R_{n_0})\rangle > b_{n_0}$. Let n be any integer greater than b_{n_1} let m_n be such that $b_{m_n} \leqslant n < b_{m_n+1}$. By definition of f, there is an m_0 such that $f(n) = 4\epsilon_{m_0}$ and $\epsilon_{m_0}\langle b_{m_n}; \max(R_{m_0})\rangle > b_{m_0}$. Since $n > n_1$, this m_0 is greater than or equal to n_0. By Lemma 3.8, for each $m+1 \geqslant m_0$, ξ_{m+1} is chosen so that $A(\nu, R_{m+1}, T, b_m, \ell_{m+1})/\langle \ell_{m+1}; r\rangle^2$ is sufficiently small to ensure

$$D(\{r^j\xi_m\} : \langle b_m; r\rangle < j \leqslant \langle b_{m+1}; r\rangle) < (\epsilon_{m+1}/10)^4.$$

By Lemma 3.9, for each m greater than or equal to m_0,

$$D(\{r^j\xi\} : \langle b_m; r\rangle < j \leqslant \langle b_{m+1}; r\rangle) < \epsilon_m \leqslant \epsilon_{m_0}.$$

Fix m so that $\langle b_m; r\rangle \leqslant \langle n; r\rangle < \langle b_{m+1}; r\rangle$. By a direct count,

$$D(\{r^j\xi\} : \langle b_{m_0}; r\rangle < j \leqslant \langle b_m; r\rangle) < \epsilon_{m_0}.$$

By Lemma 3.3,

$$D(\{r^j\xi\} : 0 \leqslant j < \langle b_m; r\rangle) < 2\epsilon_{m_0}.$$

And again by Lemma 3.3,

$$D(\{r^j\xi\} : 0 \leqslant j < \langle n; r\rangle) < 4\epsilon_{m_0} = f(n).$$

Furthermore, since $\lim_{n\to\infty} f(n) = 0$ we have $\lim_{n\to\infty} D(\{r^j\xi\} : 0 \leqslant j < \langle n; r\rangle) = 0$. Consequently, ξ is normal base r.

Consider the base s. During each stage m, the value of $D(\{s^j\xi_m\} : \langle b_m; s\rangle < j \leqslant \langle b_{m+1}; s\rangle)$ is controlled from above and from below. First, we discuss the lower bound on the discrepancy function for ξ in base s. By construction, ξ_{m+1} is obtained from ξ_m by adding a rational number whose s^{k_m}-adic expansion omits at least the digit $s^{k_m} - 1$. Further, the same digit $s^{k_m} - 1$ in base s^{k_m} was omitted every previous stage (omitting $s^k - 1$ in base s^k precludes a length k sequence of digits $s - 1$ in base s). Then, for any n such that $\langle b_m; s\rangle \leqslant n < \langle b_{m+1}; s\rangle$,

$$D(\{s^j\xi\} : 0 \leqslant j < \langle n; s\rangle) \geqslant 1/(2s^{k_m}).$$

By construction, k_m is defined so that $1/(2s^{k_m}) > g(\langle b_m; s\rangle) \geqslant g(n)$. Hence,

$$D(\{s^j\xi\} : 0 \leqslant j < \langle n; s\rangle) > g(n).$$

Now, we treat the upper bound. Let m be a stage. Let m_0 be the greatest stage less than or equal to m such that $k_{m_0} \neq k_{m_0-1}$. By construction, k_{m_0} and ℓ_{m_0} satisfy the conditions of Lemma 3.14 with input ϵ equal to $1/(4s^{k_{m_0}})$. Since $\ell_m \geqslant \ell_{m_0}$, the same holds during stage m. Consider the selection of ν during stage m. By Lemma 3.17, more than half of the eligible candidates satisfy the inequality $A(\nu, R_m, T, b_{m-1}, \ell_m)/\langle \ell_m; \max(R_m)\rangle^2 < \delta$. By Lemma 3.14, more than half the candidates satisfy

$$D(\{s^j\nu\} : 1 \leqslant j \leqslant \langle \ell_m; s\rangle) < 1/(4s^k_{m_0}).$$

Consequently, ξ_m will be defined so that

$$D(F, (\{s^j\xi_m\} : \langle b_{m-1}; s\rangle < j \leqslant \langle b_m; s\rangle))$$

is less than $1/(4s^k_{m_0})$, where F is as indicated in the construction. By Lemma 3.1,

$$D(\{s^j\xi_m\} : \langle b_{m-1}; s\rangle < j \leqslant \langle b_m; s\rangle) < 3(4s^{k_{m_0}})^{-1/2}.$$

As already argued, $\lim_{m\to\infty} \epsilon_m = 0$. Similarly, the values of $\overline{k}_m$ and the maximum element of R_m become arbitrarily large as m increases. It follows that $\lim_{m\to\infty} \ell_m = \infty$. Since g is a monotonically decreasing function and $\lim_{n\to\infty} g(n) = 0$, for every stage m there will be a later stage m_1 such that $k_{m_1} > k_m$. Thus, $\lim_{m\to\infty} D(\{s^j\xi_m\} : \langle b_{m-1}; s\rangle < j \leqslant \langle b_m; s\rangle) = 0$. It follows from Lemma 3.9, that $\lim_{m\to\infty} D(\{s^j\xi\} : \langle b_{m-1}; s\rangle < j \leqslant \langle b_m; s\rangle) = 0$, and from Lemma 3.4 that $\lim_{N\to\infty} D(\{s^j\xi\} : 0 \leqslant j < N) = 0$. Hence ξ is normal to base s. By Maxfield's Theorem, ξ is normal to every base multiplicatively dependent to s. Thus, ξ is absolutely normal.

4.6 Proof of Theorem 5

Theorem 5. *Let R be a set of bases closed under multiplicative dependence. There are real numbers normal to every base from R and not simply normal to any base in its complement. Furthermore, such a real number can be obtained computably from R.*

Let S denote the set of bases in the complement of R. Fix an enumeration of S such that every element of S appears infinitely often. The case in which 2 is an element of S requires special attention and we treat it separately.

The case $2 \notin S$. Assume that 2 is not an element of S. Fix an enumeration of S in which every element of S appears infinitely often. We define a sequence ξ_m, b_m, s_m, ϵ_m, ℓ_m, R_m and c_m. b_m is a positive integer, ϵ_m a positive rational number and R_m a finite set of bases multiplicatively independent to s_m. ξ_m is an s_m-adic rational number of precision $\langle b_m; s_m\rangle$. The real ξ will be an element of $[\xi_m, \xi_m + s_m^{-\langle b_m;s_m\rangle})$. Stage $m+1$ is devoted to extending ξ_m so that the discrepancy of the extension is below ϵ_{m+1} for the bases in R_{m+1} and so that the extension in base s_{m+1} omits the digit $s_{m+1}-1$. ℓ_m determines the length of the extension. c_m is a counter to track progress through the enumeration of S with repetitions.
Initial stage. Let $\xi_0 = 0$, $b_0 = 0$, s_0 be the least element of S, $\epsilon_0 = 1$, $\ell_0 = 0$, $R_0 = \{r_0\}$ where r_0 is the least element of R and $c_0 = 1$
Stage $m+1$. Given ξ_m of the form $\sum_{j=1}^{\langle b_m;s^{k_m}\rangle} v_j(s_m^{k_m})^{-j}$, b_m, s_m, ϵ_m, ℓ_m, R_m and c_m.

(1) If $D(\{[1-1/s_m, 1]\}, (\{s_m^j \xi_m\} : 0 \leqslant j < \langle b_m; s_m\rangle)) < (1/4)(1/s_m)$, then let $s_{m+1} = s_m$, $\epsilon_{m+1} = \epsilon_m$, $\ell_{m+1} = \ell_m$, and $R_{m+1} = R_m$.

(2) Otherwise, let $c = c_m + 1$. Let s be the cth element in the enumeration of S. Let r be the least element of R not in R_m. Let L be the least integer greater than $\max(c\,p(s_m, s)\log(\max(R_m)), \ell(R_m \cup \{r\}, s, 1, 1/c))$. If $(1/c)\langle b_m; \max(R_m)\rangle \leqslant L + p(s_m, s)$ then let s_{m+1} be s_m, let ϵ_{m+1} be ϵ_m, let ℓ_{m+1} be ℓ_m, R_{m+1} be R_m and c_{m+1} be c_m.

(3) Otherwise, let s_{m+1} be s, ϵ_{m+1} be $1/c$, ℓ_{m+1} be L, R_{m+1} be $R_m \cup \{r\}$ and c_{m+1} be c.

Let a_{m+1} be minimal such that there is an s_{m+1}-adic subinterval of $[\xi_m, \xi_m + s_m^{-\langle b_m;s_m\rangle})$ with measure $s_{m+1}^{-\langle a_{m+1};s_{m+1}\rangle}$ and the leftmost such subinterval be $[\eta_{m+1}, \eta_{m+1} + s_{m+1}^{-\langle a_{m+1};s_{m+1}\rangle})$. Let $\tilde{s}$ be $s_{m+1}-1$ if s_{m+1} is odd and be $s_{m+1}-2$ otherwise. Let T and δ be as determined in Lemma 3.8 for input $\epsilon = (\epsilon_{m+1}/10)^4$. Let ν be in $[\eta_{m+1}, \eta_{m+1} + s_{m+1}^{-\langle a_{m+1};s_{m+1}\rangle})$ such that

- $\nu = \eta_{m+1} + \displaystyle\sum_{j=1}^{\langle \ell_{m+1};s_{m+1}\rangle} w_j s_{m+1}^{-(\langle a_{m+1};s_{m+1}\rangle + j)}$, for some $(w_1, \ldots, w_{\langle \ell_{m+1};s_{m+1}\rangle})$ in $\mathcal{L}(\tilde{s}, \langle \ell_{m+1}; s_{m+1}\rangle)$
- $A(\nu, R_{m+1}, T, b_m, \ell_{m+1})/\langle \ell_{m+1}; \max(R_{m+1})\rangle^2 < \delta$

We define ξ_{m+1} to be ν and b_{m+1} to be $a_{m+1} + \ell_{m+1}$. This ends the description of stage $m+1$.
We verify that the construction succeeds. Let $m+1$ be a stage. If clause (1) or (2) applies during stage $m+1$, let m_0 be the greatest stage less than or equal to $m+1$ such that $c_{m_0} = c_{m_0+1} = \cdots = c_{m+1}$. During stage m_0, ℓ_{m_0} was chosen to satisfy the conditions to reach clause (3). Note that since $b_m > b_{m_0}$ these conditions apply to b_m in place of b_{m_0}: $(1/c_{m+1})\langle b_m; \max(R_{m_0})\rangle > \ell_{m+1} + p(s_{m_0-1}, s_{m+1})$ and ℓ_{m+1} is the maximum of $c_{m+1}p(s_{m_0-1}, s_{m+1})$ and $\ell(R_{m+1}, s_{m+1}, 1, 1/c_{m+1})$. If clause (3) applies during stage $m+1$,

then the analogous conditions hold by construction. Then, stage $m+1$ determines the subinterval $[\eta_{m+1}, \eta_{m+1} + (s_{m+1})^{-\langle a_{m+1}; s_{m+1}\rangle})$ of the interval provided at the end of stage m. Following that, it selects ν and finishes the stage. The existence of an appropriate ν is ensured by Lemma 3.17 applied to the parameters of the construction, as anticipated in the definition of the ℓ function. It follows that ξ is well defined as the limit of the ξ_m. Further, since ℓ takes only positive values, b_m is an increasing function of m.

We show that c_m goes to infinity and $\epsilon_m = 1/c_m$ goes to 0. Consider a stage $m+1$. By construction, no element of $(\{s^j_{m+1}\xi_{m+1}\} : \langle a_{m+1}; s_{m+1}\rangle < j \leqslant \langle b_{m+1}; s_{m+1}\rangle)$ is in $[1 - 1/s_{m+1}, 1]$. Further, during every subsequent stage $m_1 + 1$ with $c_{m_1+1} = c_{m+1}$, we have $a_{m_1+1} = b_{m_1}$, so no element of $(\{s^j_{m+1}\xi_{m_1+1}\} : \langle b_{m_1}; s_{m+1}\rangle < j \leqslant \langle b_{m_1+1}; s_{m+1}\rangle)$ is in $[1 - 1/s_{m+1}, 1]$. By Lemma 3.5, there will be a stage $n+1$ after $m+1$ such that $c_{n+1} = c_{m+1}$ and

$$D(\{[1 - 1/s_{m+1}, 1]\}, (\{s^j_{m+1}\xi_n\} : 0 \leqslant j < \langle b_n; s_{m+1}\rangle)) \geqslant (1/4)(1/s_{m+1}).$$

Thus, clauses (1) and (2) cannot maintain the value c_{m+1} indefinitely.

Suppose that $s \in S$. There will be infinitely many stages m such that $s = s_m$. By the above, there will be infinitely many m such that $s_m = s$ and

$$D(\{[1 - 1/s_m, 1]\}, (\{s^j_m\xi_m\} : 0 \leqslant j < \langle b_m; s_m\rangle)) \geqslant (1/4)(1/s_m).$$

Since $\xi \in [\xi_m, \xi_m + s_m^{-\langle b_m; s_m\rangle})$, the same is true for ξ in place of ξ_m. It follows that ξ is not simply normal to base s.

Suppose that $r \in R$ and $\epsilon > 0$. For all sufficiently large stages, $r \in R_{m+1}$ and $\epsilon_{m+1} < \epsilon$. Consider a sufficiently large stage $m+1$. ξ_{m+1} was defined to be ν, which was chosen so that $A(\nu, R_{m+1}, T, a_{m+1}, \ell_{m+1})/\langle \ell_{m+1}; \max(R_{m+1})\rangle^2 < \delta$. By Definition 3.16,

$$A(\nu, R_{m+1}, T, a_{m+1}, \ell_{m+1}) = \sum_{t\in T} \sum_{r\in R_{m+1}} \Big| \sum_{j=\langle a_{m+1}+1; r\rangle}^{\langle b_{m+1}; r\rangle} e(r^j t\nu)\Big|^2$$

and so $\langle \ell_{m+1}; r\rangle^{-2} \sum_{t\in T}\Big| \sum_{j=\langle a_{m+1}+1; r\rangle}^{\langle b_{m+1}; r\rangle} e(r^j t\nu)\Big|^2 < \delta$. By choice of T and δ, Lemma 3.8 ensures that

$$D(r^j\nu : \langle a_{m+1}; r\rangle < j \leqslant \langle b_{m+1}; r\rangle) < (\epsilon_{m+1}/10)^4.$$

By definition of ξ, $\xi \in [\nu, \nu + (s^{k_{m+1}}_{m+1})^{-\langle b_{m+1}; s^{k_{m+1}}_{m+1}\rangle})$. By Lemma 3.9, we conclude that

$$D(r^j\xi : \langle a_{m+1}; r\rangle < j \leqslant \langle b_{m+1}; r\rangle) < \epsilon_{m+1}.$$

By construction, $\epsilon_{m+1}\ell_{m+1}$ is greater than $\log(r)\, p(s_m, s_{m+1})$. By Lemma 3.3

$$D(r^j\xi : \langle b_m; r\rangle < j \leqslant \langle b_{m+1}; r\rangle) < 2\epsilon_{m+1} < 2\epsilon.$$

It follows that ξ is normal to base r.

The case $2 \in S$. Removing 2 and retaining all of its other powers in S maintains the condition of multiplicative independence between elements of R and S. A small alteration in our construction during the stages that ensure that ξ is not simply normal for base 4 will also ensure that ξ is not simply normal for base 2, by application of Lemma 3.13:

We change clause (2) to require that ℓ_m be sufficiently large so that Lemma 3.13 applies to conclude that more than half of the base 4 sequences of length ℓ_m have simple discrepancy greater than 1/8 in base 2. This requirement is added to the others that determine ℓ_m in the general construction. Then, while the value $s_m = 4$ is maintained, we choose ν from among these sequences and so that the condition on the value of A on ν from the general construction is also satisfied. Finally, clause (1) should be changed so that in addition to the existing condition on discrepancy in base 4 there is another condition that the simple discrepancy in base 2 is less than 1/16.

Even with these changes, ξ is well-defined. Lemma 3.13 shows that more than half of the sequences ν have simple discrepancy greater than 1/8 in base 2. Lemma 3.17 shows that at least half of them satisfy the condition on the value of A. Thus, there is an appropriate ν available. Arguing as previously, ξ is not simply normal to base 2.

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